Lecture 1: Twistor Geometry

- Let $\mathbb{M}$ be complexified Minkowski spacetime with holomorphic coordinates $x^a = (x^0, x^1, x^2, x^3)$ and holomorphic metric $ds^2 = \eta_{ab}dx^a dx^b = (dx^0)^2 - (dx^1)^2 - \cdots - (dx^3)^2$.

Lorentzian-real $\mathbb{R}^{1,3}$ sits inside $\mathbb{M}$ as the slice where $x^a \in \mathbb{R}$, and other flat real slices can be chosen by taking other reality conditions.

$\boxed{\text{Exercise}}$: what are the reality conditions for $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$ inside $\mathbb{M}$?

Thanks to the local isomorphism between $SO(4,\mathbb{C})$ and $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$, we can work with a 2-spinor formalism on $\mathbb{M}$, trading any vector/co-vector index, in the $(\frac{1}{2},\frac{1}{2})$ representation for a pair of $SL(2,\mathbb{C})$ Weyl spinor indices.

Practically, this is accomplished by contracting with the Pauli matrices: for any vector V^a, define $V^{\alpha\dot{\alpha}} := \frac{\sigma_a^{\alpha\dot{\alpha}}}{\sqrt{2}} V^a = \frac{1}{\sqrt{2}}\begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}$.

This can be done for any tensor on $\mathbb{M}$, again by repeated contraction with Pauli matrices.

Observe that $\eta_{ab}V^aV^b = 2\det(V^{\alpha\dot{\alpha}})$, so V^a is null iff $\det(V^{\alpha\dot{\alpha}}) = 0$.

This implies that $V^{\alpha\dot{\alpha}}_{\text{null}} = a^\alpha \tilde{a}^{\dot{\alpha}}$ for some spinors $a^\alpha, \tilde{a}^{\dot{\alpha}}$, and the converse.

To raise/lower $SL(2,\mathbb{C})$ spinor indices we use the natural $SL(2,\mathbb{C})$-invariant tensors: $\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{\dot{\alpha}\dot{\beta}}$, with conventions $\varepsilon^{\alpha\beta}\varepsilon_{\gamma\beta} = \delta^\alpha_\gamma$, $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = 2$

and $a_\alpha = a^\beta \varepsilon_{\beta\alpha}$, $a^\alpha = \varepsilon^{\alpha\beta} a_\beta$, etc.

<u>Example</u>: Compute the covector $V_{\alpha\dot\alpha}$, along with $V_\alpha{}^{\dot\alpha}$ and $V^\alpha{}_{\dot\alpha}$.

$$V_\alpha{}^{\dot\alpha} = V^{\beta\dot\alpha}\varepsilon_{\beta\alpha} = -\varepsilon_{\alpha\beta}V^{\beta\dot\alpha} = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -(V^1+iV^2) & -V^0+V^3 \\ V^0+V^3 & V^1-iV^2 \end{pmatrix}$$

$$V^\alpha{}_{\dot\alpha} = V^{\alpha\dot\beta}\varepsilon_{\dot\beta\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -(V^1-iV^2) & V^0+V^3 \\ -V^0+V^3 & V^1+iV^2 \end{pmatrix}$$

$$V_{\alpha\dot\alpha} = V^{\beta\dot\beta}\varepsilon_{\beta\alpha}\varepsilon_{\dot\beta\dot\alpha} = V_\alpha{}^{\dot\beta}\varepsilon_{\dot\beta\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} -(V^1+iV^2) & -V^0+V^3 \\ V^0+V^3 & V^1-iV^2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} V^0-V^3 & -(V^1+iV^2) \\ -V^1+iV^2 & V^0+V^3 \end{pmatrix}$$

This implies that the Minkowski metric in spinor variables is given by $ds^2 = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\,dx^{\alpha\dot\alpha}dx^{\beta\dot\beta}$, where $x^{\alpha\dot\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix}$

To obtain $\mathbb{R}^{1,3} \subset \mathbb{M}$, we need a reality condition on $x^{\alpha\dot\alpha}$ that corresponds to $x^a \in \mathbb{R}$. Consider the Hermitian conjugate $(x^{\alpha\dot\alpha})^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar x^0+\bar x^3 & \bar x^1-i\bar x^2 \\ \bar x^1+i\bar x^2 & \bar x^0-\bar x^3 \end{pmatrix}$; then $x^{\alpha\dot\alpha} = (x^{\alpha\dot\alpha})^\dagger$ is the desired condition, and complex conjugation interchanges the Weyl spinor representations: for $\lambda^\alpha = (a,b)$, $\tilde\omega^{\dot\alpha} = (c,d)$

$\lambda^\alpha \mapsto \bar\lambda^{\dot\alpha} = (\bar a, \bar b)$ and $\tilde\omega^{\dot\alpha} \mapsto \bar{\tilde\omega}^\alpha = (\bar c, \bar d)$.

Exercise: What are the reality conditions on $x^{\alpha\dot{\alpha}}$ needed to obtain $\mathbb{R}^4$ and $\mathbb{R}^{2,2}$? How do these act on the spinor representations?

The Twistor Correspondence: consider $\mathbb{CP}^3$, the space of all complex lines through the origin of $\mathbb{C}^4$. This can be described with homogeneous coordinates $Z^A = (Z^1, Z^2, Z^3, Z^4) \in \mathbb{C}^4$ obeying:

$$Z^A \neq 0 \quad , \quad \text{and} \quad Z^A \sim r Z^A \quad \text{for all } r \in \mathbb{C}^*.$$

The scaling condition is often called "projective scaling," and the fact that all 4 of the Z^A cannot be simultaneously zero means that $\mathbb{CP}^3$ can be charted with open sets $\mathcal{U}_i = \{Z^A \in \mathbb{C}^4 \mid Z^i \neq 0\}$ $\forall i = 1, \ldots, 4$.

On each open set, we have 3 holomorphic affine coordinates given by $\frac{Z^A}{Z^i}$, so $\mathbb{CP}^3$ is a 3-complex-dimensional space.

Let us make the arbitrary decomposition of the Z^A into two Weyl spinors of opposite chirality: $Z^A = (\mu^{\dot{\alpha}}, \lambda_\alpha)$; i.e. $\mu^{\dot{\alpha}} = (Z^1, Z^2)$, $\lambda_\alpha = (Z^3, Z^4)$

We can now relate $\mathbb{CP}^3$ to $\mathbb{M}$ by the algebraic incidence relations:

$\mu^{\dot{\alpha}} = i x^{\alpha\dot{\alpha}} \lambda_\alpha$. What is the geometric content of this relation?

First, suppose we fix some $x \in \mathbb{M}$. Then if we forget about projective scaling, $\mu^{\dot{\alpha}} = i x^{\alpha\dot{\alpha}} \lambda_\alpha$ defines a $\mathbb{C}^2 \subset \mathbb{C}^4$, and projectivising gives $\mathbb{CP}^1 \subset \mathbb{CP}^3$.

Furthermore, this relation is linear and holomorphic.

Thus, a point $x \in \mathbb{M}$ corresponds to a holomorphic, linearly embedded Riemann sphere $X \cong \mathbb{CP}^1$ inside $\mathbb{CP}^3$. We usually refer to these as "twistor lines."

Conversely, consider a fixed point $Z \in \mathbb{CP}^3$. This can be characterised as the intersection of two twistor lines, say $X, Y \subset \mathbb{CP}^3$. ④

Then by the incidence relations, $\mu^{\dot\alpha} = i x^{\alpha\dot\alpha}\lambda_\alpha$ and $\mu^{\dot\alpha} = i y^{\alpha\dot\alpha}\lambda_\alpha$ for some $x, y \in \mathbb{M}$, and thus $(x-y)^{\alpha\dot\alpha}\lambda_\alpha = 0$.

Now, as $x \neq y$ and $(x-y)^{\alpha\dot\alpha}\lambda_\alpha = \varepsilon^{\alpha\beta}(x-y)_\beta{}^{\dot\alpha}\lambda_\alpha$, this can only be true if $(x-y)^{\alpha\dot\alpha} = \lambda^\alpha\tilde\lambda^{\dot\alpha}$ for some $\tilde\lambda^{\dot\alpha}$, which is arbitrary.

Thus, $Z \in \mathbb{CP}^3$ corresponds to a $\mathbb{C}^2 \subset \mathbb{M}$ whose tangent space is spanned by vectors of the form $\lambda^\alpha\tilde\lambda^{\dot\alpha}$. These vectors are intrinsically null, so $Z \in \mathbb{CP}^3$ describes a totally null 2-plane $\alpha_Z \subset \mathbb{M}$ with $T_{\alpha_Z} = \mathrm{span}\{\lambda^\alpha\tilde\lambda^{\dot\alpha}\}$.

But we also conclude that two twistor lines intersect iff the corresponding points in $\mathbb{M}$ are null separated!

The holomorphicity and linearity of this correspondence means that we can apply our usual 3-dimensional intuition to the geometry.

Thus a twistor line can be specified by taking any two points, say $Z_1, Z_2 \in \mathbb{CP}^3$. Then by the incidence relations, $\mu_1^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_{1\alpha}$, $\mu_2^{\dot\alpha} = x^{\alpha\dot\alpha}\lambda_{2\alpha}$ for some $x \in \mathbb{M}$.

Now, consider
$$Z_1^{[A}Z_2^{B]} = \frac{1}{2}\begin{pmatrix} -(x^{\alpha\dot\alpha}x^{\beta\dot\beta} - x^{\alpha\dot\beta}x^{\beta\dot\alpha})\lambda_{1\alpha}\lambda_{2\beta} & i x^{\gamma\dot\alpha}(\lambda_{1\gamma}\lambda_{2\beta} - \lambda_{1\beta}\lambda_{2\gamma}) \\ i(\lambda_{1\alpha}\lambda_{2\gamma} - \lambda_{1\gamma}\lambda_{2\alpha})x^{\gamma\dot\beta} & \lambda_{1\alpha}\lambda_{2\beta} - \lambda_{1\beta}\lambda_{2\alpha} \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} \varepsilon^{\dot\alpha\dot\beta}x^{\alpha\dot\gamma}x^{\beta}{}_{\dot\gamma}\lambda_{1\alpha}\lambda_{2\beta} & +i x_\beta{}^{\dot\alpha}\langle\lambda_1\lambda_2\rangle \\ -i x_\alpha{}^{\dot\beta}\langle\lambda_1\lambda_2\rangle & \varepsilon_{\alpha\beta}\langle\lambda_1\lambda_2\rangle \end{pmatrix} = \frac{\langle\lambda_1\lambda_2\rangle}{2}\begin{pmatrix} \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}x^2 & i x_\beta{}^{\dot\alpha} \\ -i x_\alpha{}^{\dot\beta} & \varepsilon_{\alpha\beta} \end{pmatrix} \equiv X^{AB}.$$

So X^{AB} encodes a point in $\mathbb{M}$ plus an overall projective scale $\langle\lambda_1\lambda_2\rangle$ which tells us which points we picked on the line.

Here, we used the identity $A^{\alpha\beta} - A^{\beta\alpha} = -\varepsilon^{\alpha\beta}A^\gamma{}_\gamma$.

<u>Conformal Structure</u>: So far, nothing we've said in $\mathbb{CP}^3$ has encoded anything about $\mathbb{M}$ other than it's conformal structure (ie, null separations).

The conformal group of $\mathbb{M}$ is $SO(6,\mathbb{C}) \cong SL(4,\mathbb{C})$, so conformally-invariant objects can naturally be built using ε_{ABCD}, the $SL(4)$-invariant Levi-Civita symbol.

On $\mathbb{CP}^3$, this means that $\varepsilon_{ABCD} Z_1^A Z_2^B Z_3^C Z_4^D$ corresponds to a conformal invariant on spacetime. But this is proportional to $\varepsilon_{ABCD} X^{AB} Y^{CD}$, for $X^{AB} = Z_1^{[A} Z_2^{B]}$, $Y^{CD} = Z_3^{[C} Z_4^{D]}$, and $\varepsilon_{ABCD} X^{AB} Y^{CD} \propto \langle \lambda_1 \lambda_2 \rangle \langle \lambda_3 \lambda_4 \rangle (x-y)^2$.

So $ds^2 = \varepsilon_{ABCD} dX^{AB} dX^{CD}$ defines a conformally flat metric on $\mathbb{C}^4$; to fix the conformal scale we must make this metric well-defined projectively, which requires $ds^2 = \dfrac{\varepsilon_{ABCD} dX^{AB} dX^{CD}}{P_2(X)}$, where $P_2(X)$ obeys $P_2(rX) = r^2 P_2(X)$.

As P_2 is a homogeneous function of degree 2, it will generically have zeros, so $ds^2 = \dfrac{\varepsilon_{ABCD}\, dX^{AB} dX^{CD}}{P_2(X)}$ is a metric on $\{X^{AB} \in \mathbb{CP}^5 \,|\, X^2 = 0\} \setminus \{P_2(X) =$

with $P_2(X) = 0$ corresponding to "points at infinity."

We need to fix this P_2 to get a metric on $\mathbb{M}$ itself, and this requires introducing a new structure which breaks conformal invariance.

Define $I_{AB} := \dfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{\alpha\beta} \end{pmatrix}$, and let $P_2(X) = (I_{AB} X^{AB})^2$. Then

$P_2(X) = \dfrac{\langle \lambda_1 \lambda_2 \rangle^2}{4}$ for $X^{AB} = Z_1^{[A} Z_2^{B]}$, and $P_2(X) = 0 \Leftrightarrow$ twistor lines for which points on the line have the form $Z^A = (\mu^{\dot\alpha}, 0)$.

We can now see that this corresponds to spatial infinity, i^0, in the conformal compactification of $\mathbb{M}$. With spherical coordinates, $x^{\alpha\dot\alpha} = r\begin{pmatrix} \frac{x^0}{r}+\cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & \frac{x^0}{r}-\cos\theta \end{pmatrix}$

and $x^2 = (x^0)^2 - r^2$.

Take a twistor line $X^{AB} = Z_1^{[A} Z_2^{B]}$ which limits onto the set $P_2(x) = 0$ as $Z^A_{1,2} = (\varepsilon i x^{\alpha\dot\alpha}\lambda_{1,2\,\beta}, \varepsilon \lambda_{1,2\,\beta})$ for $\varepsilon \to 0$.

As the Z^A are homogeneous, they cannot all vanish as $\varepsilon \to 0$, so if we simultaneously take $r \to \infty$ such that $\lim\limits_{\substack{\varepsilon\to 0\\ r\to\infty}} r\varepsilon = 1$, then

$$\lim_{\substack{\varepsilon\to 0\\ r\to\infty}} X^{AB} = \frac{1}{4}\begin{pmatrix} \varepsilon^{\dot\alpha\dot\beta} & 0 \\ 0 & 0 \end{pmatrix} \equiv I^{AB}.$$

This must correspond to i^0 as we have taken $r\to\infty$, and $\varepsilon_{ABCD} I^{AB} I^{CD} = 0$, so it still describes a line in $\mathbb{CP}^3$.

Thus, $ds^2 = \dfrac{\varepsilon_{ABCD}\, dX^{AB} dX^{CD}}{(I_{AB}X^{AB})^2}$ is the Minkowski metric!

$\boxed{\text{Exercise}}$: Use the explicit formulae for X^{AB}, I_{AB} to show that $ds^2 = dx_{\alpha\dot\alpha} dx^{\alpha\dot\alpha}$, as required.

<u>Twistor Space</u>: motivated by this, we define the twistor space of $\mathbb{M}$ to be $\mathbb{PT} := \{ Z^A \in \mathbb{CP}^3 \mid \lambda_\alpha \neq 0 \} \overset{\text{open}}{\subset} \mathbb{CP}^3$. In other words, this is $\mathbb{CP}^3$ with the line "at infinity" removed.

The correspondence between $\mathbb{PT}$ and $\mathbb{M}$ can be summarized with:

$Z \in \mathbb{PT} \Leftrightarrow \alpha_Z \cong \mathbb{C}^2 \underset{\text{null}}{\subset} \mathbb{M}$

$X \cong \mathbb{CP}^1 \subset \mathbb{PT} \Leftrightarrow x \in \mathbb{M}$

$X \cap Y \neq \emptyset \Leftrightarrow x, y \in \mathbb{M}$ null separated

$I \subset \mathbb{PT} \Leftrightarrow i \in \overline{\mathbb{M}}$

$X \cap I \neq \emptyset \Leftrightarrow x \in \mathcal{I}$, conformal null infinity

Exercise: Define $\bar{Z}_A = (\bar{\lambda}_{\dot{\alpha}}, \bar{\mu}^{\alpha})$, with incidence relation $\bar{\mu}^{\alpha} = -i(x^{\alpha\dot{\alpha}})^{\dagger} \bar{\lambda}_{\dot{\alpha}}$, and $Z \cdot \bar{Z} = [\mu \bar{\lambda}] + \langle \bar{\mu} \lambda \rangle$. Show that $\mathbb{PN} = \{ Z \in \mathbb{PT} \mid Z \cdot \bar{Z} = 0 \}$ is the twistor space of $\mathbb{R}^{1,3}$, in the sense that $x \in \mathbb{R}^{1,3} \Leftrightarrow X \subset \mathbb{PN}$ and $Z \in \mathbb{PN}$ corresponds to the Lorentzian-real null vector on $\mathbb{R}^{1,3}$ $\lambda^{\alpha} \bar{\lambda}^{\dot{\alpha}}$.