Lecture 2: The Penrose Transform

A key theme in twistor theory is the ability to solve certain PDEs in terms of algebraic geometry. Here we consider the example of free massless field equations.

<u>Zero-Rest-Mass Fields</u>: consider totally symmetric spinor fields $\phi_{\alpha_1\dots\alpha_{2s}}$, $\tilde{\phi}_{\dot\alpha_1\dots\dot\alpha_{2s}}$ which are conformal densities of weight -1.

That is, under $\eta_{ab} \to \Omega^2(x)\eta_{ab} \Leftrightarrow \varepsilon_{\alpha\beta} \to \Omega\,\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta} \to \Omega\,\varepsilon_{\dot\alpha\dot\beta}$ we have

$$\phi_{\alpha_1\dots\alpha_{2s}} \to \Omega^{-1}\phi_{\alpha_1\dots\alpha_{2s}}, \quad \tilde{\phi}_{\dot\alpha_1\dots\dot\alpha_{2s}} \to \Omega^{-1}\tilde{\phi}_{\dot\alpha_1\dots\dot\alpha_{2s}}$$

We say that these objects are zero-rest-mass fields if they obey

$$\partial^{\alpha_1\dot\alpha}\phi_{\alpha_1\dots\alpha_{2s}} = 0\ , \quad \partial^{\alpha\dot\alpha_1}\tilde{\phi}_{\dot\alpha_1\dots\dot\alpha_{2s}} = 0\ , \quad \Box\phi = 0 \ (\text{for } s=0),$$

and say that:
- $\phi_{\alpha_1\dots\alpha_{2s}}$ is a z.r.m. field of helicity $-s$
- $\tilde{\phi}_{\dot\alpha_1\dots\dot\alpha_{2s}}$ " " " " " helicity $+s$

First, we observe that these are conformally-invariant equations. For instance, let $\hat{\phi}_{\alpha_1\dots\alpha_{2s}} := \Omega^{-1}\phi_{\alpha_1\dots\alpha_{2s}}$, and $\hat{\nabla}_{\alpha\dot\alpha}$ be the Levi-Civita connection of the conformally-transformed metric.

Then

$$\Omega\,\hat{\nabla}_{\beta\dot\beta}\hat{\phi}_{\alpha_1\dots\alpha_{2s}} = \Omega\,\hat{\nabla}_{\beta\dot\beta}\left(\Omega^{-1}\phi_{\alpha_1\dots\alpha_{2s}}\right) = \partial_{\beta\dot\beta}\phi_{\alpha_1\dots\alpha_{2s}} - \sum_{i=1}^{2s}\Upsilon_{\alpha_i\dot\beta}\,\phi_{\alpha_1\dots\alpha_{i-1}\beta\alpha_{i+1}\dots\alpha_{2s}} - \Upsilon_{\beta\dot\beta}\,\phi_{\alpha_1\dots\alpha_{2s}}$$

for $\Upsilon_{\alpha\dot\alpha} := \dfrac{\Omega^{-k}}{k}\partial_{\alpha\dot\alpha}\Omega^{k}$ for all $k \in \mathbb{Z}$.

Contracting both sides of this relation with $\Omega^{-2}\varepsilon^{\beta\alpha_1}\varepsilon^{\dot\beta\dot\alpha}$ gives:

$$\Omega\,\hat{\nabla}^{\alpha_1\dot\alpha}\hat{\phi}_{\alpha_1\dots\alpha_{2s}} = \Omega^{-2}\partial^{\alpha_1\dot\alpha}\phi_{\alpha_1\dots\alpha_{2s}} - \Upsilon_{\alpha_1}{}^{\dot\alpha}\phi^{\alpha_1}{}_{\alpha_2\dots\alpha_{2s}} - \Upsilon^{\alpha_1\dot\alpha}\phi_{\alpha_1\dots\alpha_{2s}}$$

$$= \Omega^{-2}\partial^{\alpha_1\dot\alpha}\phi_{\alpha_1\dots\alpha_{2s}}$$

- Thus $\hat{\nabla}^{\alpha_1\dot{\alpha}}\hat{\phi}_{\alpha_1\ldots\alpha_{2s}} = \Omega^{-3}\partial^{\alpha_1\dot{\alpha}}\phi_{\alpha_1\ldots\alpha_{2s}}$, and if $\phi_{\alpha_1\ldots\alpha_{2s}}$ is a z.r.m. on $\mathbb{M}$ then so is $\hat{\phi}_{\alpha_1\ldots\alpha_{2s}}$ on the conformally-flat spacetime.
- The proof works identically for positive helicity.
- Exercise: what happens for a massless scalar? You may find the identity for the conformal transformation of the Ricci scalar useful:

$$\hat{R} = \Omega^{-2}\left[R - 6\Omega^{-1}\Box\Omega\right].$$

The zero-rest-mass equations are just the free-field equations for massless helicity $\pm s$ linearised fields.
This is obvious for $s=0$ (massless scalar) and $s=\frac{1}{2}$ (Weyl neutrino), but perhaps less so for higher spins. Let's show this explicitly for $s=1$

<u>Helicity & Self-duality</u>:
Let $A_a = A_{\alpha\dot{\alpha}}$ be a Maxwell gauge potential, and $F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}$ be its gauge-invariant field strength.
As $F_{ab} = -F_{ba}$, we can decompose $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ into two pieces; one which is symmetric in $\alpha\leftrightarrow\beta$, and one symmetric in $\dot{\alpha}\leftrightarrow\dot{\beta}$. Defining $F_{\alpha\beta} := \frac{1}{2}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}}$ and $\tilde{F}_{\dot{\alpha}\dot{\beta}} := \frac{1}{2}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}$, this is given by

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}}F_{\alpha\beta} + \varepsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}}$$

Under Hodge duality on $\mathbb{M}$, $\frac{1}{2}\varepsilon^{abcd}F_{ab} = i\varepsilon^{\gamma\delta}\tilde{F}^{\dot{\gamma}\dot{\delta}} - i\varepsilon^{\dot{\gamma}\dot{\delta}}F^{\gamma\delta}$, so we call $\tilde{F}_{\dot{\alpha}\dot{\beta}}$ the <u>self-dual</u> and $F_{\alpha\beta}$ the <u>anti-self-dual</u> parts of the field strength, respectively.

- **Exercise**: show this, using $\epsilon^{abcd} = i\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon^{\dot\alpha\dot\delta}\epsilon^{\dot\beta\dot\gamma} - i\epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\epsilon^{\dot\alpha\dot\gamma}\epsilon^{\dot\beta\dot\delta}$
- Purely SD or ASD fields automatically satisfy the Maxwell equations, which are ~~the~~ $\partial_\beta{}^{\dot\alpha}\tilde F_{\dot\alpha\dot\beta} + \partial^\alpha{}_{\dot\beta} F_{\alpha\beta} = 0$, thanks to the Bianchi identity.

Now, consider a positive helicity photon in the spinor helicity formalism:

$a^{(+)}_{\alpha\dot\alpha} = \varepsilon^{(+)}_{\alpha\dot\alpha} e^{ik\cdot x}$, for $k^{\alpha\dot\alpha} = \kappa^\alpha \tilde\kappa^{\dot\alpha}$, $\varepsilon^{(+)}_{\alpha\dot\alpha} = \dfrac{b_\alpha \tilde\kappa_{\dot\alpha}}{\langle b\kappa\rangle}$ for some constant spinor b_α not proportional to κ_α.

Observe that $k\cdot\varepsilon^{(+)} = 0$, and changing b_α amounts to a residual gauge transformation:

$$\frac{b_\alpha \tilde\kappa_{\dot\alpha}}{\langle b\kappa\rangle} - \frac{b'_\alpha \tilde\kappa_{\dot\alpha}}{\langle b'\kappa\rangle} = \frac{\tilde\kappa_{\dot\alpha}}{\langle b\kappa\rangle\langle b'\kappa\rangle}\left(b_\alpha\langle b'\kappa\rangle - b'_\alpha\langle b\kappa\rangle\right)$$

$= \dfrac{\kappa_\alpha\tilde\kappa_{\dot\alpha}\langle b'b\rangle}{\langle b\kappa\rangle\langle b'\kappa\rangle}$, using the Schouten identity $b_\alpha\langle b'\kappa\rangle = b'_\alpha\langle b\kappa\rangle + \kappa_\alpha\langle b'b\rangle$,

implying gauge parameter $\chi = -i\dfrac{\langle b'b\rangle e^{ik\cdot x}}{\langle b\kappa\rangle\langle b'\kappa\rangle}$.

The field strength is $F_{\alpha\dot\alpha\beta\dot\beta} = \dfrac{i e^{ik\cdot x}}{\langle b\kappa\rangle}\tilde\kappa_{\dot\alpha}\tilde\kappa_{\dot\beta}\left(\kappa_\alpha b_\beta - \kappa_\beta b_\alpha\right)$

$= i e^{ik\cdot x}\tilde\kappa_{\dot\alpha}\tilde\kappa_{\dot\beta}\epsilon_{\alpha\beta}$, so $\tilde F_{\dot\alpha\dot\beta} = i\tilde\kappa_{\dot\alpha}\tilde\kappa_{\dot\beta} e^{ik\cdot x}$, $F_{\alpha\beta} = 0$.

Exercise: Show that a negative helicity photon with $\varepsilon^{(-)}_{\alpha\dot\alpha} = \dfrac{\kappa_\alpha \tilde b_{\dot\alpha}}{[\tilde b\tilde\kappa]}$ has

$\tilde F_{\dot\alpha\dot\beta} = 0$ and $F_{\alpha\beta} = i\kappa_\alpha\kappa_\beta e^{ik\cdot x}$

Observe that in both cases, $\partial^{\alpha\dot\alpha}\tilde F_{\dot\alpha\dot\beta} = 0$ (positive helicity) and $\partial^{\alpha\dot\alpha}F_{\alpha\beta} = 0$ (negative helicity); this is a consequence of the Bianchi identity.

- The Penrose transform: We can use twistor theory to generate solutions to the z.r.m. equations!
- Let $\mathcal{O}(k)$ denote the sheaf of locally holomorphic functions on $\mathbb{PT}$, homogenous of degree $k \in \mathbb{Z}$. This is fancy terminology for $f(Z)$ which obey $f(rZ) = r^k f(Z)$ for all $r \in \mathbb{C}^*$.
- Now, consider 1-forms on $\mathbb{PT}$ which only point in the anti-holomorphic directions. We call these $(0,1)$-forms, and denote them by $f \in \Omega^{0,1}(\mathbb{PT})$, which means that $f(Z) = f_{\bar{A}}(Z)\, d\bar{Z}^{\bar{A}}$.

Let $\Omega^{0,1}(\mathbb{PT}, \mathcal{O}(k))$ denote the space of $(0,1)$-forms on $\mathbb{PT}$ which are homogenous of weight k: $f(rZ, \bar{r}\bar{Z}) = r^k f(Z, \bar{Z})$.

We say that some $f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(k))$ is holomorphic, or $\bar{\partial}$-closed, if $\bar{\partial} f = 0$ in $\Omega^{0,2}(\mathbb{PT}, \mathcal{O}(k))$, where $\bar{\partial} = d\bar{Z}^{\bar{A}} \dfrac{\partial}{\partial \bar{Z}^{\bar{A}}}$ sends $\Omega^{p,q}(\mathbb{PT}) \to \Omega^{p,q+1}(\mathbb{PT})$

and obeys $\bar{\partial}^2 = 0$.

Let $f = \bar{\partial} g$ for some $g \in \Omega^{0}(\mathbb{PT}, \mathcal{O}(k))$; then $\bar{\partial} f = 0$ automatically as $\bar{\partial}^2 = 0$

The cohomology group $H^{0,1}(\mathbb{PT}, \mathcal{O}(k)) = \dfrac{\{f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(k)) \mid \bar{\partial} f = 0\}}{\{f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(k)) \mid f = \bar{\partial} g\}}$ is essentially

the set of holomorphic $(0,1)$-forms of weight k, modulo the "trivial" ones that are $\bar{\partial}$-exact.

Theorem (Penrose; Eastwood-Penrose-Wells): $\{$z.r.m. fields on $\mathbb{M}$ of helicity $h\}$

$$\cong H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2)).$$

- Providing a rigorous proof of this statement is a bit tricky, particularly the part of showing that every z.r.m. arises in this way.
- However, we can get a bit of understanding by considering the integral formulae; for $f \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))$:

$$\phi_{\alpha_1 \cdots \alpha_{2|h|}}(x) = \int_X \langle \lambda d\lambda \rangle \wedge \lambda_{\alpha_1} \cdots \lambda_{\alpha_{2|h|}} f|_X$$

for $f(Z)|_X = f(ix^{\alpha\dot{\alpha}}\lambda_\alpha, \lambda_\alpha)$

$$\tilde{\phi}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2h}}(x) = \int_X \langle \lambda d\lambda \rangle \wedge \frac{\partial}{\partial \mu^{\dot{\alpha}_1}} \cdots \frac{\partial}{\partial \mu^{\dot{\alpha}_{2h}}} f \Big|_X$$

$$\phi(x) = \int_X \langle \lambda d\lambda \rangle \wedge f|_X$$

$\boxed{\text{Exercise}}$: explain why these formulae make sense, ie, why are the integrals well-defined on the Rieman sphere $X \cong \mathbb{CP}^1$?

Let's consider the negative helicity case for simplicity. Using the incidence relations, $\partial^{\alpha_1\dot{\alpha}} \phi_{\alpha_1 \cdots \alpha_{2|h|}} = \int_X \langle \lambda d\lambda \rangle \wedge \lambda_{\alpha_1} \cdots \lambda_{\alpha_{2|h|}} \left(i \lambda^{\alpha_1} \frac{\partial f}{\partial \mu_{\dot{\alpha}}}\Big|_X - i \bar{\lambda}^{\alpha_1} \frac{\partial f}{\partial \bar{\mu}_{\dot{\alpha}}}\Big|_X \right)$. The first term vanishes as $\lambda_{\alpha_1}\lambda^{\alpha_1} = 0$, and the second term vanishes by the holomorphy of f (ie, $\bar{\partial} f = 0$), so $\partial^{\alpha_1\dot{\alpha}} \phi_{\alpha_1 \cdots \alpha_{2|h|}} = 0$, as desired.

Similar arguments hold for $h \geq 0$, and observe that all formulae are compatible with the "gauge invariance" $f \to f + \bar{\partial} g$, which does not alter the resulting z.r.m.

The Penrose transform means that any $f \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2h-2))$ gives rise to a z.r.m., and every z.r.m. arises in this way.

In practice, there is usually some particular representation of the z.r.m. we are interested in, and the goal is then to cook up the corresponding twistor representative.

- **Example:** Consider positive/negative helicity photons in a momentum eigenstate representation.
- On $\mathbb{M}$, we saw that these corresponded to the z.r.m.s

$\phi_{\alpha\beta} = \kappa_\alpha \kappa_\beta e^{ik\cdot x}$ and $\tilde{\phi}_{\dot\alpha\dot\beta} = \tilde{\kappa}_{\dot\alpha} \tilde{\kappa}_{\dot\beta} e^{ik\cdot x}$, for on-shell momentum $k^{\alpha\dot\alpha} = \kappa^\alpha \tilde{\kappa}^{\dot\alpha}$

- On $\mathbb{PT}$, these should correspond to some $a^{(-)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$ and $a^{(+)} \in H^{0,1}(\mathbb{PT}, \mathcal{O})$, respectively.
- Aside: let $z \in \mathbb{C}$ and consider $\bar\delta(z) := \frac{1}{2\pi i} d\bar z \frac{\partial}{\partial \bar z}\left(\frac{1}{z}\right) = \frac{1}{2\pi i} \bar\partial\left(\frac{1}{z}\right)$

To see how this behaves as a distribution, consider integration against a holomorphic function $f(z)$ on some $D \subset \mathbb{C}$, $\partial D = \Gamma$.

$$\int_D dz \wedge \bar\delta(z) f(z) = \frac{1}{2\pi i}\int_D dz \wedge \bar\partial\left(\frac{1}{z}\right) f(z) = \frac{1}{2\pi i}\int_D \bar\partial\left(\frac{dz\, f(z)}{z}\right) = \frac{1}{2\pi i}\oint_\Gamma dz \frac{f(z)}{z} = f(0)$$

by Cauchy's theorem.

So $\bar\delta(z)$ acts as a sort of (0,1)-form Dirac delta function: $\bar\delta(z) = \delta(\mathrm{Re}\, z)\delta(\mathrm{Im}\, z) d\bar z$, what mathematicians would call a (0,1)-distribution.

Now, consider $a^{(-)}(Z) = \left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)^3 \bar\delta(\langle\lambda\kappa\rangle) \exp\left(\frac{[\mu\tilde\kappa]\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)$. It is easy to see that this is homogeneous of weight -4, and a (0,1)-form on $\mathbb{PT}$.

Furthermore, $\bar\partial a^{(-)} = \bar\partial\left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)^3 \wedge \bar\delta(\langle\lambda\kappa\rangle) \exp\left(\frac{[\mu\tilde\kappa]\langle b\kappa\rangle}{\langle b\lambda\rangle}\right) + \left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)^3 \bar\partial\bar\delta(\langle\lambda\kappa\rangle) \exp\left(\frac{[\mu\tilde\kappa]\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)$

$$= \left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)^3 \bar\delta(\langle\lambda\kappa\rangle) \wedge \bar\partial\left(\frac{[\mu\tilde\kappa]\langle b\kappa\rangle}{\langle b\lambda\rangle}\right) \exp\left(\frac{[\mu\tilde\kappa]\langle b\kappa\rangle}{\langle b\lambda\rangle}\right).$$

- However, each of these terms is proportional to $d\bar{\lambda}^{\dot\alpha} \wedge d\bar{\lambda}^{\dot\beta} = 0$, as there is only one projective degree of freedom in λ_α on $\mathbb{PT}$ (as $\lambda_\alpha \neq 0$).
- Thus $\bar\partial a^{(-)} = 0$, and it is similarly straightforward to argue that $a^{(-)}$ cannot be written as $\bar\partial f$ for some $f \in \Omega^0(\mathbb{PT}, \mathcal{O}(-4))$, so $a^{(-)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$
- Furthermore, it may appear that $a^{(-)}$ depends on some constant reference spinor b_α. However,

$$\frac{\partial a^{(-)}}{\partial b^\alpha} \propto \frac{\partial}{\partial b^\alpha}\left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right) = \frac{\kappa_\alpha \langle b\lambda\rangle - \lambda_\alpha \langle b\kappa\rangle}{\langle b\lambda\rangle^2} = \frac{\langle \kappa\lambda\rangle b_\alpha}{\langle b\lambda\rangle^2}$$

using the Schouten identity, and this vanishes on the support of $\bar\delta(\langle\lambda\kappa\rangle)$, so $a^{(-)}$ doesn't actually depend on b_α.
- At last, we can evaluate $\displaystyle\int_X \langle\lambda d\lambda\rangle \wedge \lambda_\alpha \lambda_\beta \, a^{(-)}\Big|_X$

$$= \int D\lambda \wedge \lambda_\alpha \lambda_\beta \left(\frac{\langle b\kappa\rangle}{\langle b\lambda\rangle}\right)^3 \bar\delta(\langle\lambda\kappa\rangle) \exp\left[\frac{i x^{\alpha\dot\alpha} \lambda_\alpha \tilde\kappa_{\dot\alpha} \langle b\kappa\rangle}{\langle b\lambda\rangle}\right] = \kappa_\alpha \kappa_\beta e^{ikx}, \text{ as desired.}$$

Exercise: Run through similar steps to construct $a^{(+)} \in H^{0,1}(\mathbb{PT}, \mathcal{O})$ for a momentum eigenstate, and evaluate $\displaystyle\int_X \langle\lambda d\lambda\rangle \wedge \frac{\partial^2 a^{(+)}}{\partial\mu^{\dot\alpha} \partial\mu^{\dot\beta}}\Big|_X$