Lecture 3: Self-dual Gauge Theory

SD Yang-Mills fields: let $A_{\alpha\dot\alpha}$ be a gauge connection taking values in the Lie algebra $\mathfrak{g}$ of some gauge group G

The field strength of the connection $D_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + A_{\alpha\dot\alpha}$ is

$$F_{\alpha\dot\alpha\beta\dot\beta} = \partial_{\alpha\dot\alpha}A_{\beta\dot\beta} - \partial_{\beta\dot\beta}A_{\alpha\dot\alpha} + [A_{\alpha\dot\alpha}, A_{\beta\dot\beta}] = [D_{\alpha\dot\alpha}, D_{\beta\dot\beta}]$$

Non-abelian gauge transformations are given by matrix-valued functions $h(x)$ such that $A_{\alpha\dot\alpha} \to h^{-1}\partial_{\alpha\dot\alpha}h + h^{-1}A_{\alpha\dot\alpha}h$, under which $F_{\alpha\dot\alpha\beta\dot\beta} \to h^{-1}F_{\alpha\dot\alpha\beta\dot\beta}h$.

Just like the abelian case, $F_{\alpha\dot\alpha\beta\dot\beta}$ can be decomposed into SD and ASD parts: $F_{\alpha\dot\alpha\beta\dot\beta} = \varepsilon_{\alpha\beta}\tilde{F}_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}F_{\alpha\beta}$, now valued in $\mathfrak{g}$

The field strength automatically obeys the Bianchi identity:

$$D_\beta{}^{\dot\alpha}\tilde{F}_{\dot\alpha\dot\beta} - D^\alpha{}_{\dot\beta}F_{\alpha\beta} = 0.$$

The Yang-Mills equations are $D^{\alpha\dot\alpha}F_{\alpha\dot\alpha\beta\dot\beta} = D_\beta{}^{\dot\alpha}\tilde{F}_{\dot\alpha\dot\beta} + D^\alpha{}_{\dot\beta}F_{\alpha\beta} = 0$, so SD gauge fields ($F_{\alpha\beta}=0$) automatically solve the YM equations!

The Ward Correspondence: the equations $F_{\alpha\beta}=0$ are a set of 3 Lie-algebra-valued, non-linear PDEs - seems hard to solve!

Twistor theory provides a way to generate all solutions to these equations in terms of unconstrained holomorphic data on twistor space.

The basic result is due to Ward, from 1977.

<u>Theorem</u>: there is a 1:1 correspondence between:

i) SD YM fields on $\mathbb{M}$ with gauge group $GL(N,\mathbb{C})$, and

ii) Holomorphic vector bundles $E \to \mathbb{PT}$ of rank N, which are topologically trivial on restriction to every twistor line.

Let's now prove this theorem, starting with the (ii) $\Rightarrow$ (i) direction.

$E \to \mathbb{PT}$ holomorphic implies the existence of a partial connection $\bar{D}: \Omega^{p,q}(\mathbb{PT}, E) \to \Omega^{p,q+1}(\mathbb{PT}, E)$ obeying $\bar{D}^2 = 0$. Indeed, holomorphic sections of E are precisely those annihilated by $\bar{D}$.

The assumption that $E|_x$ is topologically trivial, combined with $\bar{D}^2 = 0$, implies that there will generically exist a holomorphic trivialization.

This means that there exists a "holomorphic frame" $H(x,\lambda,\bar{\lambda}): E|_x \to \mathbb{C}^N$ such that $H^{-1} \bar{D}|_x H = \bar{\partial}|_x$, the standard complex structure on $\mathbb{CP}^1$.

This in turn implies that $\bar{D}|_x H = 0$. Solutions of this equation are unique up to residual gauge transformations $H \to H h$, where $\bar{\partial}|_x h(x,\lambda,\bar{\lambda}) = 0$.

Now, h is homogeneous of degree zero on $\mathbb{CP}^1$ (albeit matrix-valued), so if it is globally holomorphic, it is in fact constant and $h = h(x)$ only.

Using the fact that $\lambda^\alpha \partial_{\alpha\dot{\alpha}}$ annihilates objects pulled back to twistor lines from $\mathbb{PT}$ and $\bar{D}|_x H = 0$, it follows that $\bar{\partial}|_x (H^{-1} \lambda^\alpha \partial_{\alpha\dot{\alpha}} H) = 0$.

$\boxed{\text{Exercise}}$: show this. You can use a local representation of $\bar{D} = \bar{\partial} + a$ for $a \in \Omega^{0,1}(\mathbb{PT}, \mathrm{End}\, E)$, and recall that for a matrix-valued function f, derivatives obey $df^{-1} = -f^{-1} df\, f^{-1}$.

This means that $H^{-1}\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}H$ is a globally holomorphic, $\mathfrak{gl}_N$-valued function of homogeneity $+1$ on $\mathbb{CP}^1$. By an extension of Liouville's theorem, this means that $H^{-1}\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}H = \lambda^{\alpha}A_{\alpha\dot{\alpha}}(x)$ for $A_{\alpha\dot{\alpha}}$ valued in $\mathfrak{gl}_N$. ③

Under $H(x,\lambda,\bar{\lambda}) \to H(x,\lambda,\bar{\lambda})h(x)$, it follows that $A_{\alpha\dot{\alpha}} \to h^{-1}\partial_{\alpha\dot{\alpha}}h + h^{-1}A_{\alpha\dot{\alpha}}h$, so $A_{\alpha\dot{\alpha}}$ is a $GL(N,\mathbb{C})$ gauge connection on M.

$\boxed{\text{Exercise}}$: show this.

$H^{-1}\lambda^{\alpha}\partial_{\alpha\dot{\alpha}}H = \lambda^{\alpha}A_{\alpha\dot{\alpha}} \Leftrightarrow \lambda^{\alpha}D_{\alpha\dot{\alpha}}H^{-1} = 0$, which is 2 matrix-valued equations for a single matrix-valued function, which must have non-trivial solutions.

The Frobenius integrability condition is $[\lambda^{\alpha}D_{\alpha\dot{\alpha}}, \lambda^{\beta}D_{\beta\dot{\beta}}] = \varepsilon_{\dot{\alpha}\dot{\beta}}\lambda^{\alpha}\lambda^{\beta}F_{\alpha\beta} = 0$ $\Leftrightarrow F_{\alpha\beta} = 0$, as desired.

Now, consider the converse: (i) $\Rightarrow$ (ii). Given a SD $GL(N,\mathbb{C})$ gauge field on M, we have $F_{ab} = \varepsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}}$.

$\forall Z \in \mathbb{PT}$, it follows that $F_{ab}|_{\alpha_Z} = \lambda^{\alpha}\tilde{a}^{\dot{\alpha}}\lambda^{\beta}\tilde{b}^{\dot{\beta}}\varepsilon_{\alpha\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} = 0$, so the SD gaugefield is trivial on restriction to each α-plane in M.

Thus $\{s(x) \text{ valued in } \mathbb{C}^N \mid D_{\alpha\dot{\alpha}}s|_{\alpha_Z}\} \cong \{\text{constant functions}\} \cong \mathbb{C}^N$. This defines a fibre $E|_Z \cong \mathbb{C}^N$ for all $Z \in \mathbb{PT}$.

As this construction is holomorphic, $E \to \mathbb{PT}$ is a rank N holomorphic vector bundle. Furthermore, since fibres are defined by local, constant functions valued in the fundamental representation, $E|_X$ will be topologically trivial.

This completes the proof.

To use this practically, one must determine H explicitly for a given SD gauge field; in general this is difficult as H is defined implicitly by $\bar{D}|_x H = 0$. However, there are special cases where we can be very explicit! (4)

<u>The Twisted Photon</u>: Consider a SD Maxwell field, for which $E \to \mathbb{PT}$ is simply a line bundle.

Locally, write $\bar{D} = \bar{\partial} + a$, for $a \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O})$. Then the holomorphicity condition is just $\bar{D}^2 = \bar{\partial} a = 0 \Leftrightarrow a \in H^{0,1}(\mathbb{PT}, \mathcal{O})$.

Now, $\bar{D}|_x = \bar{\partial}|_x + a|_x$, with $a|_x \in H^{0,1}(\mathbb{CP}^1, \mathcal{O})$, but $H^{0,1}(\mathbb{CP}^1, \mathcal{O}) = \emptyset$, which means that $\exists\, g(x, \lambda, \bar{\lambda}) \in \Omega^0(\mathbb{CP}^1, \mathcal{O})$ such that $a|_x = \bar{\partial}|_x g$.

$\boxed{\text{Exercise}}$: Show that $H^{0,1}(\mathbb{CP}^1, \mathcal{O}) = \{0\}$. First, try to do this explicitly using homogeneous coordinates, then show it using the Riemann-Roch theorem for line bundles: $\dim H^0(\Sigma, \mathcal{L}) - \dim H^1(\Sigma, \mathcal{L}^{-1}) = \deg(\mathcal{L}) + 1 - g$, for Σ a genus g Riemann surface and $\mathcal{L} \to \Sigma$ a holomorphic line bundle.

Thus $\bar{D}|_x H = 0$ is solved by $H(x, \lambda, \bar{\lambda}) = \exp[-g(x, \lambda, \bar{\lambda})]$, and $H^{-1} \lambda^\alpha \partial_{\alpha\dot{\alpha}} H$
$= -\lambda^\alpha \partial_{\alpha\dot{\alpha}} g = \lambda^\alpha A_{\alpha\dot{\alpha}}(x)$.

By fixing a gauge $\sigma^\alpha \tilde{\sigma}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} = 0$, this can be solved explicitly in terms of the twistor data: $A_{\alpha\dot{\alpha}}(x) = -\dfrac{o_\alpha}{2\pi} \displaystyle\int_x \frac{D\lambda'}{\langle o \lambda' \rangle} \frac{\partial a}{\partial \mu^{\dot{\alpha}}}\Big|_x$.

To check this, we just compute $\lambda^\alpha A_{\alpha\dot{\alpha}} = -\dfrac{\langle \lambda o \rangle}{2\pi} \displaystyle\int \frac{D\lambda'}{\langle o\lambda' \rangle} \frac{\partial a}{\partial \mu}\Big|_x$

$$= \frac{\langle o\lambda \rangle}{2\pi} \int \frac{D\lambda'\, \lambda^\alpha \lambda'_\alpha}{\langle \lambda\lambda' \rangle \langle o\lambda' \rangle} \frac{\partial a}{\partial \mu^{\dot{\alpha}}}\Big|_x = \frac{-i\langle o\lambda \rangle \lambda^\alpha \partial_{\alpha\dot{\alpha}}}{2\pi} \int_x \frac{D\lambda'}{\langle \lambda\lambda' \rangle \langle o\lambda' \rangle} \bar{\partial}|_x g(x, \lambda', \bar{\lambda}')$$

$$= -\langle o\lambda \rangle \lambda^\alpha \partial_{\alpha\dot{\alpha}} \int \frac{D\lambda'}{\langle o\lambda' \rangle} g(x, \lambda', \bar{\lambda}')\, \bar{\delta}(\langle \lambda\lambda' \rangle) = -\lambda^\alpha \partial_{\alpha\dot{\alpha}} g(x, \lambda, \lambda'),$$ as required.

• Exercise: Use this formula to prove that the Maxwell tensor of $A_{\alpha\dot{\alpha}}$ is self-dual.

Exercise: Compute $g(x,\lambda,\bar{\lambda})$ for a positive helicity momentum eigenstate

$$A_{\alpha\dot{\alpha}}(x) = \frac{o_\alpha \tilde{k}_{\dot{\alpha}}}{\langle o k \rangle} e^{ik\cdot x}, \quad k_\alpha \tilde{k}_{\dot{\alpha}} = k_{\alpha\dot{\alpha}}.$$

• Background-Coupled Penrose Transform: for a general SD gauge field, the machinery of the Penrose transform carries over to describe background coupled gluons.

The chirality of the background means that there is now an asymmetry between the definitions of positive and negative helicity.

Naively, want to say that $a^{(\pm)}_{\alpha\dot{\alpha}}$ defined by $f^{(\pm)}_{ab} = D_{[a} a^{(\pm)}_{b]}$ being purely SD/ASD. However, a gauge transformation $D_a \gamma$, for some Lie algebra-valued γ, leads to a curvature proportional to $[D_a, D_b \gamma] = \varepsilon_{\alpha\beta}[\tilde{F}_{\dot{\alpha}\dot{\beta}}, \gamma]$.

So it is not gauge invariant to say that a negative helicity gluon in the background has $f_{\dot{\alpha}\dot{\beta}} = 0$, although we can say that a positive helicity gluon has $f_{\alpha\beta} = 0$.

In other words, $\{+\text{hel. gluons on SD background}\} = \{a_{\alpha\dot{\alpha}} \text{ on } \mathbb{M} \mid f_{\alpha\beta} = D_{(\alpha}{}^{\dot{\alpha}} a_{\beta)\dot{\alpha}} = 0\}$

$\{-\text{hel. gluons on SD background}\} = \{a_{\alpha\dot{\alpha}} \text{ on } \mathbb{M} \mid D^{\alpha\dot{\alpha}} f_{\alpha\beta} = 0\}$

Now, on $\mathbb{PT}$ we can define cohomology with respect to $\bar{D}$, since $\bar{D}^2 = 0$.

Consider $H^{0,1}_{\bar{D}}(\mathbb{PT}, \mathcal{O}\otimes \mathrm{End}\, E)$, and take some $a(z)$ in this cohomology group.

It follows that $H^{-1} a|_x H \in H^{0,1}_{\bar{\partial}}(\mathbb{P}^1, \mathcal{O}\otimes \mathfrak{g}) = \emptyset$, so $H^{-1} a|_x H = \bar{\partial}|_x j(x,\lambda,\bar{\lambda})$ for some $j \in \Omega^0(\mathbb{PS}, \mathcal{O}\otimes\mathfrak{g})$.

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Now, $\lambda^\alpha D_{\alpha\dot\alpha}(H^{-1}a|_x H) = H^{-1}(\lambda^\alpha \partial_{\alpha\dot\alpha} a|_x)H = 0$

This follows as $\lambda^\alpha D_{\alpha\dot\alpha}(H^{-1}a|_x H) = \lambda^\alpha[A_{\alpha\dot\alpha}, H^{-1}a|_x H] - H^{-1}\lambda^\alpha(\partial_{\alpha\dot\alpha}H)H^{-1}a|_x H + H^{-1}(\lambda^\alpha\partial_{\alpha\dot\alpha}a|_x)H$
$+ H^{-1}a|_x \lambda^\alpha \partial_{\alpha\dot\alpha}H = \lambda^\alpha[A_{\alpha\dot\alpha}, H^{-1}a|_x H] - \lambda^\alpha A_{\alpha\dot\alpha}H^{-1}a|_x H + H^{-1}(\lambda^\alpha\partial_{\alpha\dot\alpha}a|_x)H + H^{-1}a|_x H\lambda^\alpha A_{\alpha\dot\alpha}$

$= H^{-1}(\lambda^\alpha\partial_{\alpha\dot\alpha}a|_x)H$, using $H^{-1}\lambda^\alpha\partial_{\alpha\dot\alpha}H = \lambda^\alpha A_{\alpha\dot\alpha}$

Similarly $\lambda^\alpha D_{\alpha\dot\alpha}\bar\partial|_x j = \bar\partial|_x(\lambda^\alpha D_{\alpha\dot\alpha}j)$, so $\bar\partial|_x(\lambda^\alpha D_{\alpha\dot\alpha}j) = 0$, and by our usual extension of Liouville's theorem, $\lambda^\alpha D_{\alpha\dot\alpha}j = \lambda^\alpha a^{(+)}_{\alpha\dot\alpha}(x)$

Then $\lambda^\beta D_\beta{}^{\dot\alpha}(\lambda^\alpha D_{\alpha\dot\alpha}j) = \lambda^\alpha\lambda^\beta F_{\alpha\beta}j = 0$, and $\lambda^\beta D_\beta{}^{\dot\alpha}(\lambda^\alpha a^{(+)}_{\alpha\dot\alpha}) = \lambda^\alpha\lambda^\beta f^{(+)}_{\alpha\beta}$

$\Rightarrow f^{(+)}_{\alpha\beta} = 0$, as desired for a positive helicity gluon.

For negative helicity, consider $b \in H^{0,1}_{\bar\partial}(\mathbb{PT}, \mathcal{O}(-4)\otimes \mathrm{End}\,E)$. Then $H^{-1}b|_x H$ is valued in $H^{0,1}_{\bar\partial}(\mathbb{P}^1, \mathcal{O}(-4)\otimes\mathfrak{g})$, and we can form $f_{\alpha\beta}(x) = \int_X D\lambda \wedge \lambda_\alpha\lambda_\beta H^{-1}b|_x H$

This obeys $D^{\alpha\dot\beta}f_{\alpha\beta} = \int_X D\lambda\wedge\lambda_\beta\lambda_\alpha D^{\alpha\dot\alpha}(H^{-1}b|_x H) = \int D\lambda\wedge\lambda_\beta H^{-1}(\lambda_\alpha\partial^{\alpha\dot\alpha}b|_x)H = 0$

as required for a negative helicity gluon.