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Symmetries of asymptotically flat spacetimes

Overall, the lectures have <u>two main goals</u>:

- Introduce you with asymptotic symmetries of gravity in a way which makes it clear that this is not (only) a feature of a particular coordinate system (BMS) but a property of a class of spacetime.

- Get you aquainted with enough tools and ideas to prepare you reading the (very vast) litterature on the subject and learn the rest by yourself.

Plan of the lectures

Lecture 1: null geodesic congruence

Lecture 2: Conformal compactification of Minkowski space

Lecture 3: Asymptotically flat spacetime (1):

geometry of null infinity and BMS

Lecture 4: Asymptotically flat spacetimes (2):

BMS coordinates and asymptotic symmetries.

Lecture 1: null geodesic congruence

This lecture has two purposes:

On the one hand this will serve as a historical introduction, on the other hand it will introduce ideas and tools which are of general interest in general relativity.

I will treat

- null geodesic congruence and (some of) their properties
- null tetrad and Newman-Penrose coefficients
- Sachs peeling

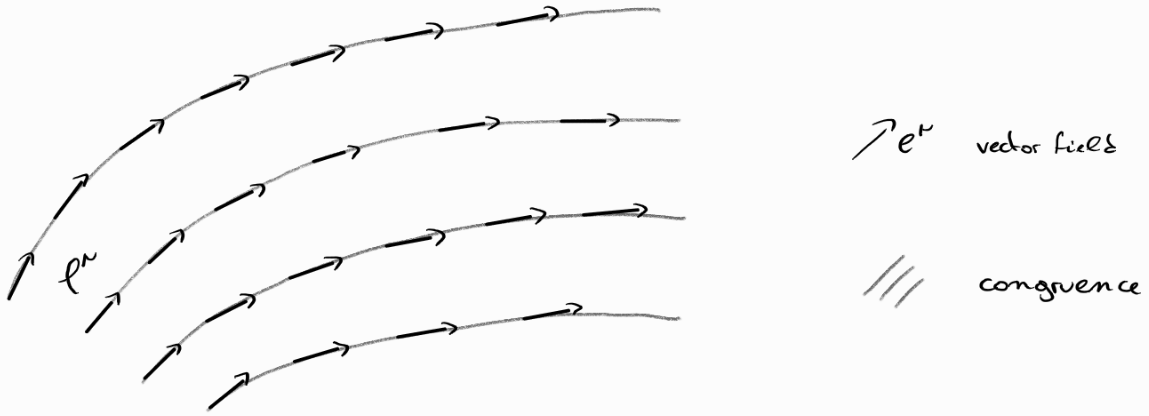
Let $(M, g_{\mu\nu})$ be a (four dimensional, Lorentzian, of signature $(-+++)$) manifold.

Definition:

Let ℓ^μ be a nowhere vanishing vector field on M.

The congruence generated by ℓ^μ is the set of integral curves for the flow of ℓ^μ.

In other words this is the set of trajectories tangential to ℓ^μ.



NB: ℓ^μ and $f(x)\,\ell^\mu$ generate the same congruence : the rescaling only change the "speed of the flow".

- A null congruence is a congruence generated by a null vector field: $\ell^\mu \ell^\nu g_{\mu\nu} = 0$
- A geodesic congruence - - - - - - covariantly constant vector field : $\ell^\nu \nabla_\nu \ell^\mu = 0$

From now on we consider a null geodesic congruence generated by ℓ^μ.

Historically, (Pirani, Trautman, Sachs ...) the first ideas to characterize the presence of gravitationnal radiations involved the investigation of the behavior of the gravitational field, i.e. the Weyl tensor, as one follows a null geodesic congruence to infinity.

$\Longrightarrow$ see more on this by the end of the lecture

A crucial step in the understanding of the asymptotics of the gravitational field was the idea to consider "hypersurface orthogonal null geodesic congruence" (Bondi, van der Burg, Metzner, Sachs)

$\Longrightarrow$ a precise definition is coming soon.

Associated coordinate system:

Proposition

Consider a null geodesic congruence generated by ℓ^μ

One can always choose (at least locally) a coordinate system $(u, r, y^{A\in 1,2})$

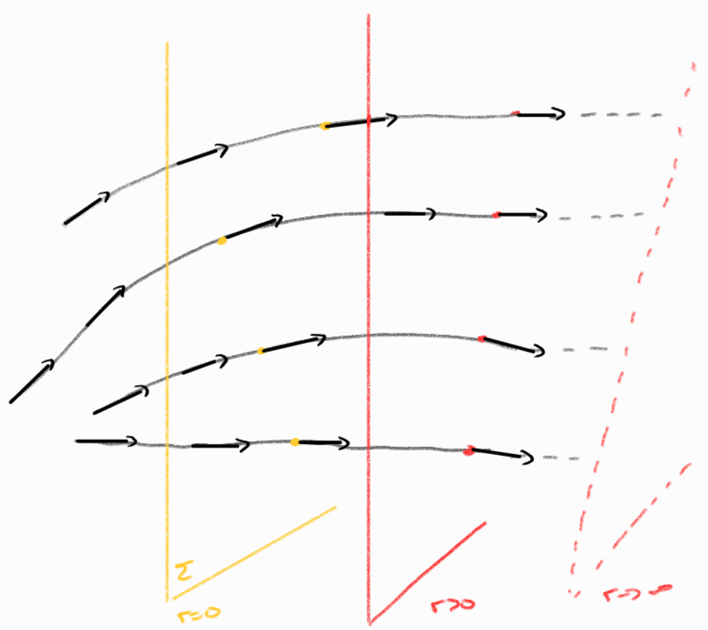
could mean $y^1 = x$, $y^2 = y$ or $y^1 = z = x+iy$, $y^2 = \bar{z} = x - iy$

such that $\ell = \frac{\partial}{\partial r}$ ← r is the affine parameter along the geodesic congruence

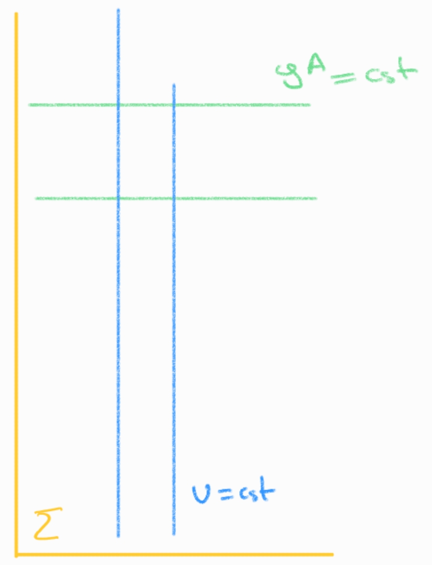
$$\boxed{ds^2 = V(du)^2 + 2\,du\,dr + \mathcal{H}_{AB}\left(dy^A - V^A du\right)\left(dy^B - V^B du\right) + D_A\, dy^A dr}$$

with $\frac{\partial}{\partial r} D_A = 0$ and $V, \mathcal{H}_{AB}, V^A$ are functions of (u, r, y^A)

null congruence



Coordinates on Σ



Extra

proof

1) Locally, one can always take a hypersurface Σ transverse to ℓ^μ

2) One can always choose $(u, y^A) \in \mathbb{R}^3$ a local coordinate system on Σ

3) Define r as the affine time along the congruence starting at Σ (i.e. $r|_\Sigma = 0$)

$\Rightarrow \ell = \frac{\partial}{\partial r}$

4) In full generality:

$$ds^2 = V (du)^2 + B\, 2 du\, dr + A (dr^2)$$
$$+ \mathcal{H}_{AB} (dy^A - \upsilon^A du)(dy^B - \upsilon^B du)$$
$$+ D_A\, dy^A dr$$

Then $\ell^2 = 0 \Rightarrow A = 0$

$\nabla_\ell \ell = 0 \Rightarrow \partial_r D_A = 0$ and $\partial_r B = 0$

5) Finally one can always make a change of coordinates

$(u, y^A) \mapsto (\hat{u}(u, y^A), \hat{y}^A = y^A)$ such that $\frac{\partial \hat{u}}{\partial u} = B(u, y^A) \Leftrightarrow d\hat{u} = B\, du + (\dots) dy^A$

and this gives $\hat{B} = 1 \Leftrightarrow \ell^\mu g_{\mu\nu} dx^\nu = d\hat{u} + \hat{D}_A d\hat{y}^A$

Remark:

In the previous coordinate system one has: $\ell^\nu g_{\mu\nu} dx^\nu = du + D_A dy^A$

Definition

A null geodesic congruence is hypersurface orthogonal if there exists a function $u : M \to \mathbb{R}$ such that:

$$\ell^\mu g_{\mu\nu} = (du)_\nu \iff \ell^\mu = g^{\mu\nu}\partial_\nu u$$

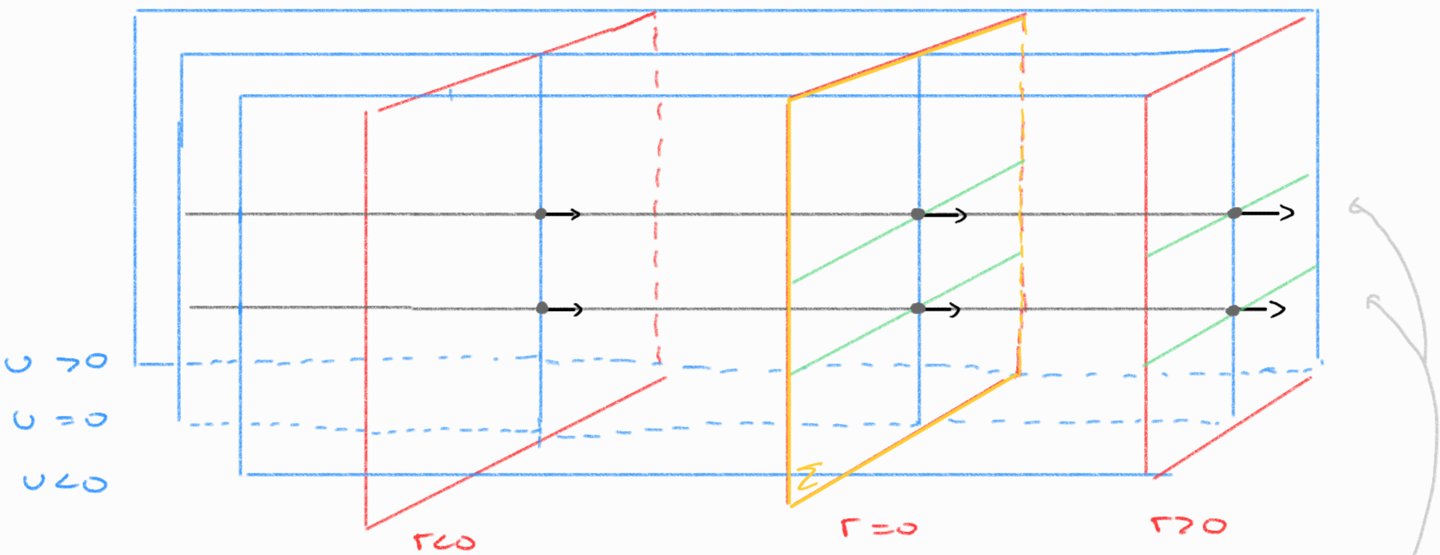
NB

- ℓ^μ really is orthogonal to the hypersurfaces $u = cst$:

if X^μ is any vector field along the hypersurfaces $u=cst$, then $X^\mu \ell^\nu g_{\mu\nu} = X^\mu (du)_\mu = 0$.

- The hypersurfaces $u=cst$ are null hypersurfaces: ℓ^μ is both tangential and orthogonal to these hypersurfaces.

hypersurface orthogonal null geodesic congruence



The two null geodesics (in grey) are along the same $u=0$ hypersurfaces but at different coordinates y^A

⚠ $\ell^\mu = \partial_r$ is both along, and orthogonal to, the $u=cst$ hypersurfaces. ⚠

Associated coordinate system:

Proposition

A null geodesic congruence is hypersurface orthogonal if and only if one can choose locally a coordinate system (u, r, y^A) such that and

$\ell = \frac{\partial}{\partial r}$

$$ds^2 = V(du)^2 + 2du\,dr + \mathcal{H}_{AB}\left(dy^A - \mathcal{V}^A du\right)\left(dy^B - \mathcal{V}^B du\right)$$

"Newman-Unti" coordinates

proof

Same proof as for the previous coordinate system but taking u to be the function given by the definition.

⚠ Not every null geodesic congruence is hypersurface orthogonal

(see next page for some characterisation.)

⚠ If one takes another "time" r along the geodesics (i.e. not the affine parametrization) then $\ell = e^{-\beta}\partial_r$ and

$$ds^2 = V(du)^2 + \underset{\neq 1}{e^{\beta}}\, 2du\,dr + \mathcal{H}_{AB}\left(dy^A - \mathcal{V}^A du\right)\left(dy^B - \mathcal{V}^B du\right)$$

"BMS-like" coordinates

Extra

Proposition

A null geodesic congruence is hypersurface orthogonal if and only if

$$\ell_{[\mu} \partial_\nu \ell_{\rho]} = 0$$

proof

This is Frobenius theorem:
define $\Theta := \ell_\mu dx^\mu$ then

$$\exists u \text{ s.t. } \Theta = du \iff \Theta \wedge d\Theta = 0 \iff \ell_{[\mu} \partial_\nu \ell_{\rho]} = 0$$

Proposition

The previous condition is also equivalent to the requirement that the congruence is "not twisting" (see later for the definition).

proof

exercise: prove that the vanishing of the twist ω_{AB} of a null geodesic congruence is equivalent to the condition $\ell_{[\mu} \partial_\nu \ell_{\rho]} = 0$.

Basics of the Newman-Penrose Formalisme

Null tetrad

at every point x of M $(\ell, m, \bar{m}, n)$ form a basis of $T_x M$

A null tetrad is a frame field $(\ell^\mu, m^\mu, \bar{m}^\mu, n^\mu)$ such that the only non zero contractions with respect to the metric are:

⚠ ℓ^μ and n^μ are real vector fields but $m^\mu = \frac{m_1^\mu + i m_2^\mu}{\sqrt{2}}$, $\bar{m}^\mu = \frac{m_1^\mu - i m_2^\mu}{\sqrt{2}}$ are complex

$$\ell^\mu n^\nu g_{\mu\nu} = -1 \quad , \quad m^\mu \bar{m}^\nu g_{\mu\nu} = 1$$

NB:
- in particular $\ell^2 = n^2 = m^2 = \bar{m}^2 = 0$
- this implies that $m_\alpha^\mu m_\beta^\nu g_{\mu\nu} = \delta_{\alpha\beta}$ $\quad \alpha, \beta \in 1,2$

Proposition : null tetrad adapted to a null geodesic congruence.

Let us consider a null geodesic congruence generated by ℓ^μ.

Let $(\ell^\mu, m_\alpha^\mu, n^\mu)$ be an adapted null tetrad (it is "adapted" because the first vector generates the congruence)

Then any other adapted null tetrad $(\ell^\mu, \hat{m}^\mu, \hat{\bar{m}}^\mu, \hat{n}^\mu)$ must be of the form

$$\boxed{\begin{aligned} \hat{m}^\mu &= e^{i\varphi(x)} \left(m^\mu + f(x) \ell^\mu \right) \\ \hat{n}^\mu &= n^\mu + f(x) \bar{m}^\mu + \bar{f}(x) m^\mu + \tfrac{1}{2} |f|^2 \ell^\mu \end{aligned}}$$

$f(x) \in \mathbb{C}$
$\varphi(x) \in \mathbb{C}$

"m^μ is defined up to addition of ℓ^μ"

"n^μ - uniquely fixed by choosing m^μ"

proof : exercise

Corollary : NP coefficients

Let $(M, g_{\mu\nu})$ be a vacuum solution of Einstein's equation with vanishing cosmological constant.

Let ℓ^μ be the generator of a null geodesic congruence.

Then $\boxed{\Psi_0 := R_{\mu\nu\rho\sigma}\, \ell^\mu m^\nu \ell^\rho m^\sigma} \in \mathbb{C}$

is an invariant of the congruence ← i.e. it does not depend on the choice of m^μ

- if $\Psi_0 = 0$ then $\Psi_1 := R_{\mu\nu\rho\sigma}\, \ell^\mu m^\nu \ell^\rho n^\sigma \in \mathbb{C}$

  becomes an invariant of the congruence

- if $\Psi_0 = \Psi_1 = 0$ then $\Psi_2 := R_{\mu\nu\rho\sigma}\, \ell^\mu m^\nu \bar{m}^\rho n^\sigma \in \mathbb{C}$

  becomes an invariant of the congruence

- if $\Psi_0 = \Psi_1 = \Psi_2 = 0$ then $\Psi_3 := R_{\mu\nu\rho\sigma}\, \ell^\mu n^\nu \bar{m}^\rho n^\sigma \in \mathbb{C}$

  becomes an invariant of the congruence

- if $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$ then $\Psi_4 := R_{\mu\nu\rho\sigma}\, n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma \in \mathbb{C}$

  becomes an invariant of the congruence

(• if $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0$ then $g_{\mu\nu}$ is flat)

proof

exercise hint: use Bianchi identity $R_{\mu[\nu\rho\sigma]} = 0$

and $R_{\mu\nu\rho\sigma}(m^\mu \bar{m}^\rho - \ell^\mu n^\rho) = R_{\mu\nu\rho\sigma}\, g^{\mu\rho} = R_{\nu\sigma} = 0$

Remarks

- these are called the Newman-Penrose coefficients
- if one does not impose the Einstein condition the NP coefficients are defined in terms of the Weyl tensor.

Sachs peeling

Sach's idea (1961) was to characterize an asymptotically flat spacetime by requiring

$$\Psi_0 \underset{r\to+\infty}{\sim} r^{-5}\,\Psi_0^\circ + O(r^{-6})$$

$$\Psi_1 \underset{r\to+\infty}{\sim} r^{-4}\,\Psi_1^\circ + O(r^{-5})$$

$$\Psi_2 \underset{r\to+\infty}{\sim} r^{-3}\,\Psi_2^\circ + O(r^{-4})$$

$$\Psi_3 \underset{r\to+\infty}{\sim} r^{-2}\,\Psi_3^\circ + O(r^{-3})$$

$$\Psi_4 \underset{r\to+\infty}{\sim} r^{-1}\,\Psi_4^\circ + O(r^{-2})$$

Note that this makes invariant sense because the NP coefficients go to zero in the correct order.

$\Longrightarrow$ Because all other NP coefficients vanish asymptotically $\boxed{\Psi_4^\circ \text{ is an asymptotic invariant}}$

It characterizes the asymptotic presence of gravitational waves.

Corollary : exponsion, sheor ond twist

the deviation tensor

$$\boxed{B_{\alpha\beta} := m_\alpha{}^\nu m_\beta{}^\sigma \nabla_\nu \ell_\sigma} \qquad \alpha, \beta \in 1,2$$

is an invariant of a null geodesic congruence.

i.e. it does not depend on the choice of m'

it decomposes into :

$$\boxed{B_{\alpha\beta} = \frac{\Theta}{2}\delta_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta}}$$

Exponsion

Sheor

twist

where

$$\Theta := \delta^{\alpha\beta} B_{\alpha\beta}\,, \quad \sigma_{\alpha\beta} := B_{(\alpha\beta)}\Big|_{\substack{\text{trace}\\ \text{-free}}}\,, \quad \omega_{\alpha\beta} = B_{[\alpha\beta]}\,.$$

Proposition

For a hypersurface orthogonal null geodesic congruence and in the coordinate system (u, r, y^A) such that

$$ds^2 = V(du)^2 + 2du\,dr + \mathcal{H}_{AB}\,(dy^A - \mathcal{V}^A du)(dy^B - \mathcal{V}^B du)$$

"Newman-Unti" coordinates

- one can take $\ell = \frac{\partial}{\partial r}$, $m_\alpha = E_\alpha{}^A \frac{\partial}{\partial y^A}$ where $E_\alpha^A E_\beta^B \mathcal{H}_{AB} = \delta_{\alpha\beta}$

- $B_{\alpha\beta} = E_\alpha^A E_\beta^B \left(\frac{1}{2}\partial_r \mathcal{H}_{AB}\right) \implies \omega_{\alpha\beta} = 0 \quad \Theta = \frac{\mathcal{H}^{AB}\partial_r \mathcal{H}_{AB}}{2}\,, \quad \sigma_{\alpha\beta} = \frac{1}{2}E_\alpha^A E_\beta^B (\partial_r \mathcal{H}_{AB})\big|_{TF}$

proof

direct computation