Yannick Herfray

Lecture 2:
Conformal compactification of Minkowski space

The goal of this lecture is to treat in details the conformal compactification of Minkowski space and introduce on the way essential notions that will generalize in the asymptotically flat case.

The conformal compactification of Minkowski space

Minkowski

$$\tilde{ds}^2 = -dt^2 + dr^2 + r^2 d\Sigma^2$$ here $t, r \in \mathbb{R}$ $r > 0$

$$(d\Sigma)^2 = \frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}$$ metric on S^2

Proposition

- if one takes $u = t - r$ and $v = t + r$ then

$$\tilde{ds}^2 = -du\,dv + \frac{1}{4}(v-u)^2 d\Sigma^2$$ here $u, v \in \mathbb{R}$ with $v - u = 2r > 0$

- then, if one takes $u = \tan U$, $v = \tan V$ then

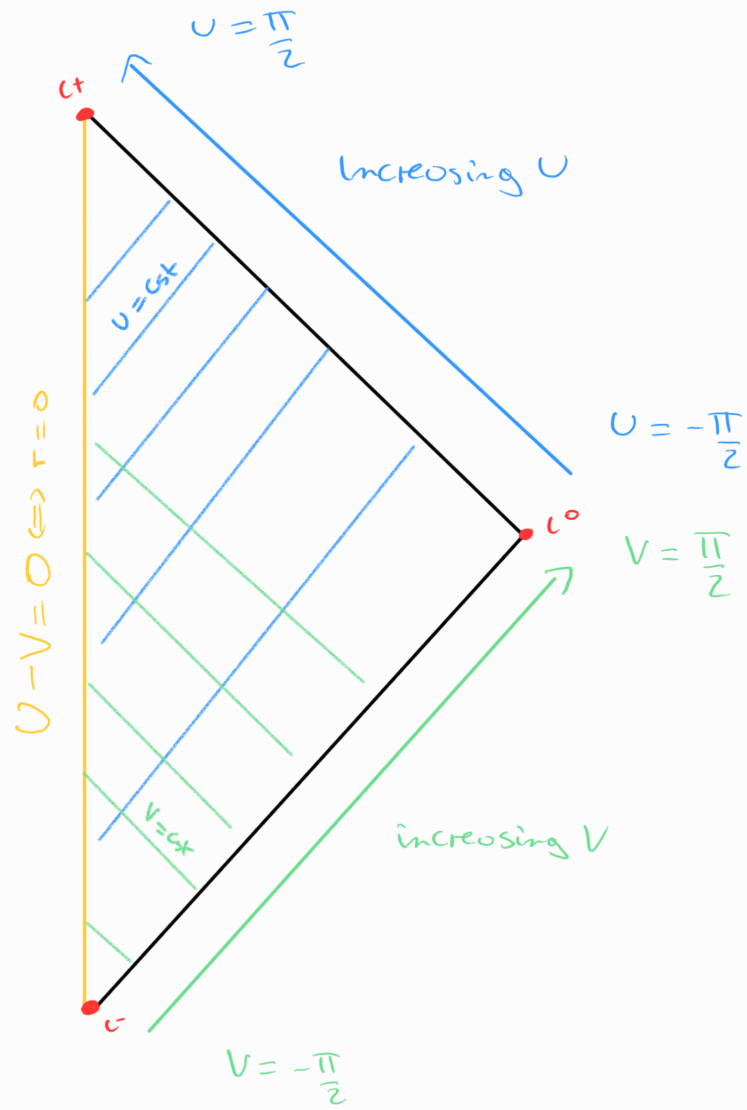
$$\tilde{ds}^2 = \frac{1}{4\cos^2 U \cos^2 V}\left[-4\,dU\,dV + \sin^2(V-U)\,d\Sigma^2\right]$$ here $U, V \in (-\frac{\pi}{2}, \frac{\pi}{2})$ with $V - U \geq 0$

$\frac{1}{4\cos^2 U \cos^2 V} = \frac{1}{\Omega^2}$

$\left[-4\,dU\,dV + \sin^2(V-U)\,d\Sigma^2\right] = ds^2$ "Unphysical metric"

$\Omega = 2\cos U \cos V$ "Conformal scale"

Conformal diagram of Minkowski



Crucial remarks

1) In this picture we "added a boundary at infinity":

The physical metric: $d\tilde{s}^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$ is only defined in the interior = Minkowski

The unphysical metric: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (\Omega^2 \tilde{g}_{\mu\nu}) dx^\mu dx^\nu$ extends to the boundary.

$\Longrightarrow$ **Conformal compactification**

Crucial remarks

2) The boundary is <u>not</u> a manifold but rather is made of five different pieces (each of which is a manifold):

- future timelike infinity i^+ : $U = \frac{\pi}{2}, V = \frac{\pi}{2}$

  the induced metric is $ds^2\big|_{i^+} = 0 \implies$ point

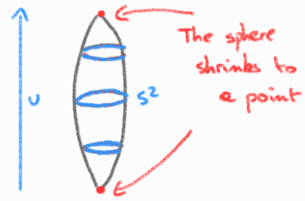
- future null infinity $\mathscr{I}^+$: $U \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $V = \frac{\pi}{2}$

It is a 3d manifold of topology $S^2 \times \mathbb{R}$ with coordinates $(U, z, \bar{z})$

the induced metric is $ds^2\big|_{\mathscr{I}^+} = \boxed{0\,(dU)^2} + \cos^2(U)\left(\frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}\right)$

The metric is degenerate along the U direction!

The radius of the 2-sphere tends to zero as one goes to $U = \pm\frac{\pi}{2}$:



- Space like infinity i^0 : $U = -\frac{\pi}{2}$, $V = \frac{\pi}{2}$

  the induced metric is $ds^2\big|_{i^0} = 0 \implies$ point

- past null infinity $\mathscr{I}^-$: $U = -\frac{\pi}{2}$, $V \in (-\frac{\pi}{2}, \frac{\pi}{2})$

It is a 3d manifold of topology $S^2 \times \mathbb{R}$ with coordinates $(V, z, \bar{z})$

the induced metric is $ds^2\big|_{\mathscr{I}^-} = \boxed{0\,(dV)^2} + \cos^2(V)\left(\frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}\right)$

The metric is degenerate along the V direction!

- past timelike infinity i^- : $U = -\frac{\pi}{2}, V = -\frac{\pi}{2}$

  the induced metric is $ds^2\big|_{i^-} = 0 \implies$ point

Crucial remarks

3) From the previous point, one sees that
(we note $(\hat{\mathbb{M}}, \hat{g}_{\mu\nu})$ Minkowski space)

i) There exists $\mathbb{M} = \hat{\mathbb{M}} \sqcup \mathscr{I}^+ \sqcup \mathscr{I}^-$ a manifold with boundary $\partial \hat{\mathbb{M}} = \mathscr{I}^+ \sqcup \mathscr{I}^-$

ii) There exists $(\Omega, g_{\mu\nu})$ a scalar field and metric on $\mathbb{M}$ such that $\hat{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$ on the interior $\hat{\mathbb{M}}$ (= Minkowski)

iii) On the boundary $\mathscr{I}^+ \sqcup \mathscr{I}^-$ $\Omega = 0$ but $\nabla_\mu \Omega \neq 0$.

Immediately generalizes to the notion of asymptotically flat spacetime.

4) There is nothing unique abaut the unphysical metric and the conformal scale: $(\Omega, g_{\mu\nu})$.

We could have taken $\left(\hat{\Omega} = f(x)\,\Omega,\ \hat{g}_{\mu\nu} = f^2(x)\, g_{\mu\nu}\right)$ for any $f \in \mathcal{C}^\infty(\mathbb{M})$

and this would give the same physical metric:

$$\hat{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu} = \frac{1}{\hat{\Omega}^2} \hat{g}_{\mu\nu}$$

⚠

$\Longrightarrow$ On the manifold with boundary we only have

- A conformal class of metric $[g_{\mu\nu}]$ $\quad g_{\mu\nu} \sim f^2(x)\, g_{\mu\nu}$
- A conformal density $[\Omega]$ $\quad (\Omega, g_{\mu\nu}) \sim (f(x)\,\Omega, f^2(x)\, g_{\mu\nu})$

Crucial remarks

5) In particular the induced (degenerate) metric on $\mathscr{I}^+$ is only defined up to

$$h_{ab}\,dx^a dx^b = 0\,dU^2 + \cos^2(U)\left(\frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}\right) \qquad x^a = (U, z, \bar{z})$$

$$\sim \hat{h}_{ab}\,dx^a dx^b = f^2(U,z,\bar{z})\; h_{ab}\,dx^a dx^b$$

$\Longrightarrow$ We can take $\hat{h}_{ab}\,dx^a dx^b = 0\,dU^2 + \left(\frac{4\,dz\,d\bar{z}}{(1-|z|^2)^2}\right)$

such that the sphere has constant radius 1.

(see the BMS coordinates later)

6) In particular the normal to $\mathscr{I}^+$ $\qquad \Omega = 2\cos U \cos V$

$$n^a \frac{\partial}{\partial x^a} = g^{\mu\nu}\partial_\nu(\Omega)\frac{\partial}{\partial x^\mu}\Big|_{\mathscr{I}^+} = -\frac{1}{2}\left(\partial_U \Omega\right)\Big|_{V=\frac{\pi}{2}}\frac{\partial}{\partial V} - \frac{1}{2}\left(\partial_V \Omega\right)\Big|_{V=\frac{\pi}{2}}\frac{\partial}{\partial U} = \cos U \frac{\partial}{\partial U}$$

is only defined up to

$$\left(n^a\frac{\partial}{\partial x^a},\; h_{ab}\,dx^a dx^b\right) \sim \left(\hat{n} = f(x)^{-1} n^a \frac{\partial}{\partial x^a},\; \hat{h} = f^2(x)\, h_{ab}\,dx^a dx^b\right)$$

This is the "universal (or Carrollian) structure" of null infinity

NB: taking $f = \cos^{-1}(U)$ gives $\left(\hat{n} = \cos^2(U)\frac{\partial}{\partial U} = \partial_u \;,\; \hat{h} = \frac{4\,dz\,d\bar{z}}{(1+|z|^2)^2}\right)$ $\qquad u = \tan U$

BMS coordinate system

Minkowski

$$\tilde{ds}^2 = -dt^2 + dr^2 + r^2 d\Sigma^2$$

here $t, r \in \mathbb{R}$ $r > 0$

$$(d\Sigma)^2 = \frac{4\, dz\, d\bar{z}}{(1 - |z|^2)^2}$$ metric on S^2

Proposition

- if one takes $u = t - r$ and $\Omega = r^{-1}$

$$\tilde{ds}^2 = -du^2 + 2du\,dr + r^2 d\Sigma^2$$

here $u, r \in \mathbb{R}$ with $r > 0$

$$= \frac{1}{\Omega^2}\left(-\Omega^2 du^2 - 2du\,d\Omega + d\Sigma^2\right)$$

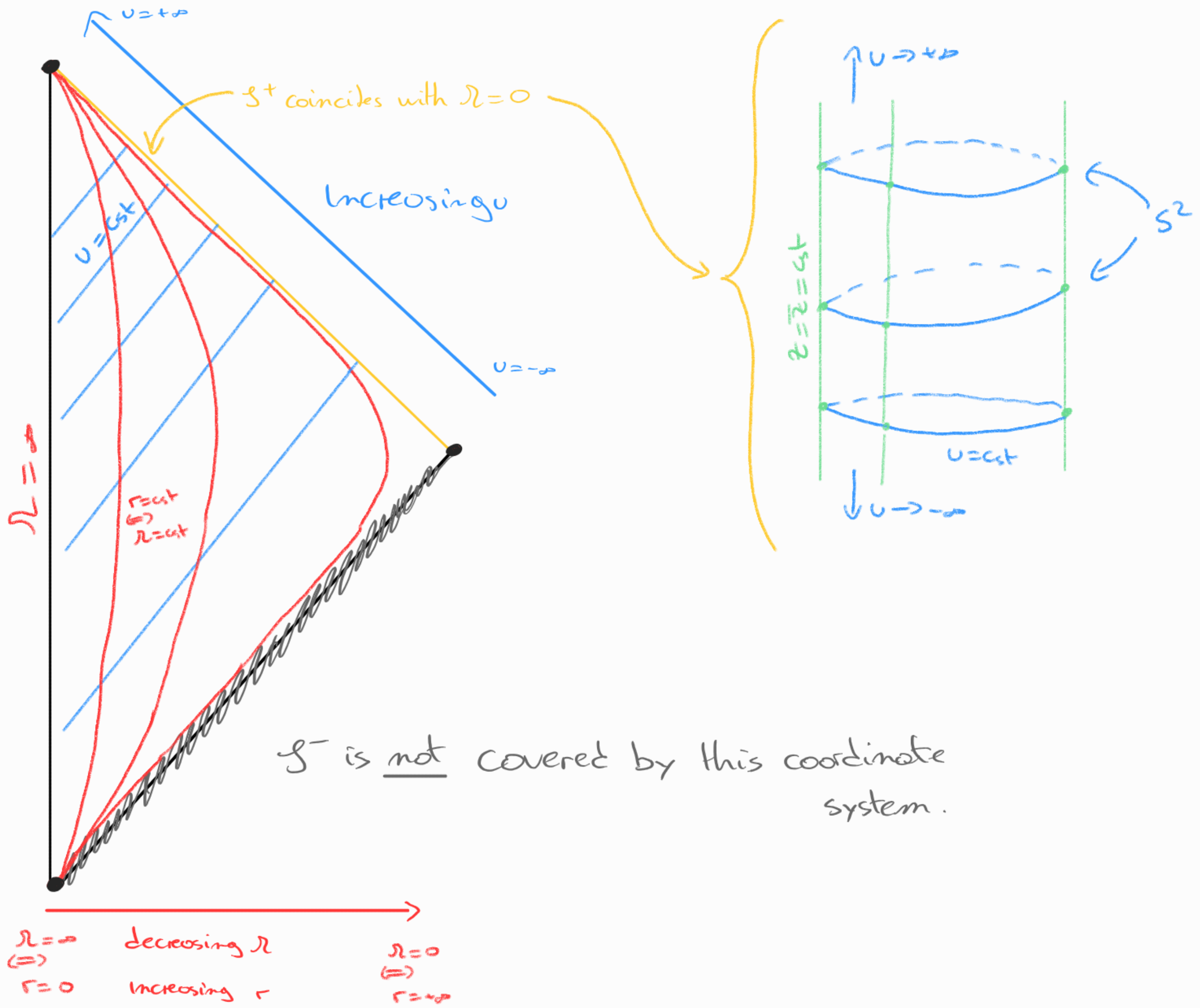
here $u, \Omega \in \mathbb{R}$ with $\Omega \geq 0$

$\Omega = \frac{1}{r}$ ds^2

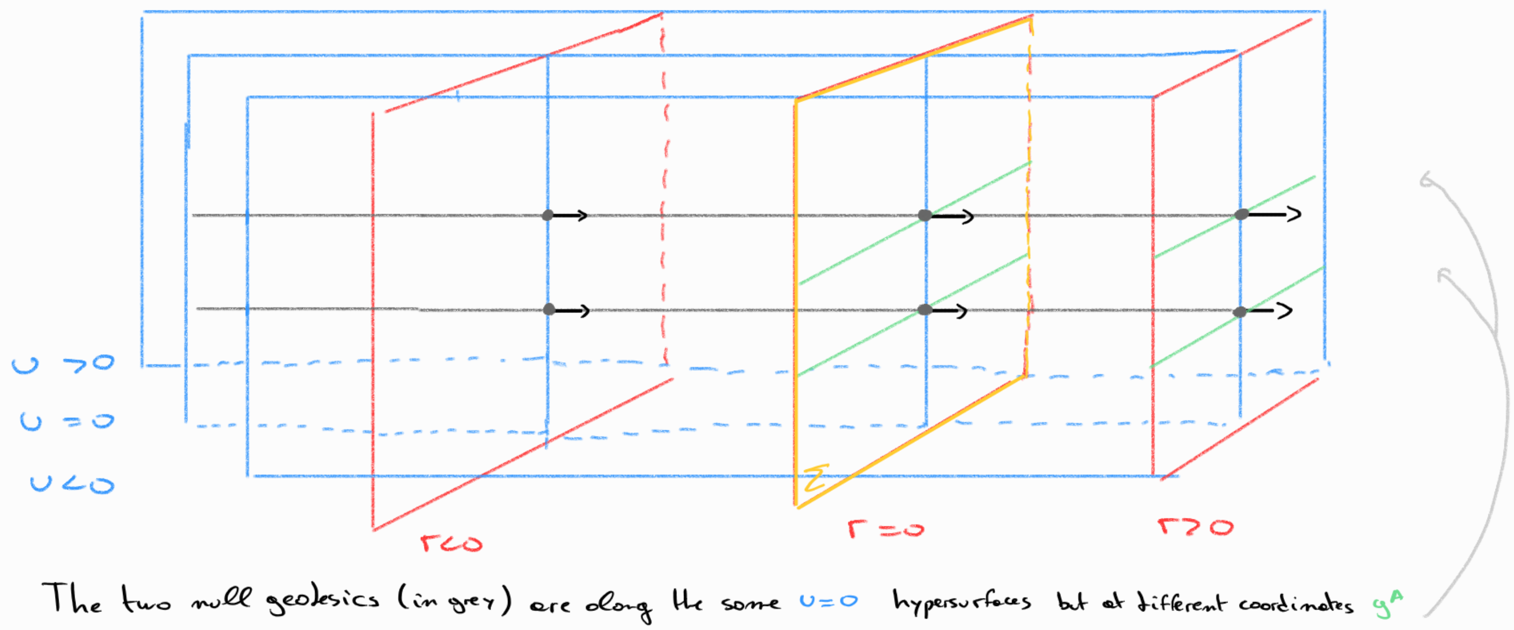
By the results of the previous lecture this defines a (hypersurface orthonormal) null geodesic congruence with

$\Theta = r$, $\sigma_{\alpha\beta} = 0$, $\omega_{\alpha\beta} = 0$.

BMS coordinates



BMS coordinates directly relate to a (hypersurface orthogonal) null geodesic congruence starting at $r=\infty \Leftrightarrow \Omega=0$. Compare with the previous lecture:



The two null geodesics (in grey) are along the same $u=0$ hypersurfaces but at different coordinates y^A

Universal/Carrollian structure of null infinity

in BMS coordinates

- Induced degenerate metric

$$h_{ab}\, dx^a dx^b := g_{\mu\nu}\, dx^\mu dx^\nu \Big|_{\mathscr{I}^+} = 0\,(du)^2 + \frac{4\, dz d\bar{z}}{(1+|z|^2)^2}$$

- Normal vector field

$$n^a \frac{\partial}{\partial x^a} := g^{\mu\nu} \nabla_\mu \Omega \frac{\partial}{\partial x^\nu}\Big|_{\mathscr{I}^+} = \frac{\partial}{\partial u}$$