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Lecture 3:
Asymptotically flat spacetime (1)
Null infinity and the BMS group

Goal of these last two lectures:

$\boxed{(1)}$ Investigate some consequences of Penrose's definition for "asymptotically flat spacetimes":

- Universal (a.k.a. Carrollian) structure of null infinity
- BMS group
- Penrose's peeling

(2) Show the equivalence with BMS coordinates and relate to the first lecture

Asymptotically flat spacetime

Definition (Penrose 1962)

A spacetime $(\hat{M}, \hat{g}_{\mu\nu})$ is asymptotically flat if

i) There exists a manifold $M = \hat{M} \cup \mathscr{I}$ with boundary $\mathscr{I} = \partial M$

ii) There exists $(\Omega, g_{\mu\nu})$ a scalar field and metric on M

such that $\hat{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$ on the interior of M $(= \hat{M})$

iii) On the boundary $\mathscr{I}$, $\Omega\big|_{\mathscr{I}} = 0$, $\nabla_\mu \Omega\big|_{\mathscr{I}} \neq 0$.

iv) $\hat{g}_{\mu\nu}$ satisfies Einstein's equations

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\,\hat{g}_{\mu\nu} = \hat{T}_{\mu\nu} \quad \text{with} \quad \hat{T}_{\mu\nu} = \mathcal{O}(\Omega^2)$$

NB: As discussed in the previous lectures the unphysical metric $g_{\mu\nu}$ and the scalar field Ω are only defined up to weyl rescaling

$$(\Omega, g_{\mu\nu}) \sim (\lambda(x)\Omega,\ \lambda^2(x) g_{\mu\nu})$$

In this lecture we will derive a certain number of consequences of Penrose's définition

Universal structure of null infinity

In what follows $x^\mu = (\Omega, x^a)$ are coordinates on M
x^a – – – $\mathscr{I}$

Proposition

- the normal at $\mathscr{I}$ $\quad n := g^{\mu\nu}\nabla_\nu \Omega \dfrac{\partial}{\partial x^\mu}\Big|_{\mathscr{I}}$

is necessarily null $\quad n^\mu n^\nu g_{\mu\nu} = 0$

$\Longrightarrow$ this implies that $\quad n^\mu \nabla_\mu \Omega = 0$

i.e. the normal is tangential to $\mathscr{I}$: $\quad n = n^a \dfrac{\partial}{\partial x^a}$

- the induced metric $\quad h_{ab}\,dx^a dx^b := g_{\mu\nu}\,dx^\mu dx^\nu\Big|_{\mathscr{I}}$

is degenerate of signature $(0,+,+)$,

it satisfies

$$n^a h_{ab} = 0 \quad , \quad \mathcal{L}_n h_{ab} = \alpha\, h_{ab}$$

proof see below

proof

We first recall the definition of the Schoten tensor (in 4d)

$$P_{\mu\nu} := \frac{1}{2}\left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu}\right)$$

It takes a direct computation to check that the Schoten tensor

$P_{\mu\nu}$ of the unphysical metric $\hat{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$

$\hat{P}_{\mu\nu}$ – – physical – $g_{\mu\nu}$

are related by

$$\hat{P}_{\mu\nu} = P_{\mu\nu} + \Omega^{-1} \nabla_\mu \nabla_\nu \Omega - \frac{1}{2} \Omega^{-2} (\nabla_\rho \Omega)(\nabla^\rho \Omega) g_{\mu\nu}$$

Now Einstein's equation on the physical metric imply that

$\hat{P}_{\mu\nu}\Big|_{\mathscr{I}^+} = 0$ and in particular cannot diverge

This means that $n^\mu n^\nu g_{\mu\nu}\Big|_{\mathscr{I}} = \nabla_\rho \Omega \nabla^\rho \Omega\Big|_{\mathscr{I}} = 0$

This proves the first point of the proposition

and $\nabla_\mu n_\nu\Big|_{\mathscr{I}} = (\nabla_\rho n^\rho) \frac{1}{4} g_{\mu\nu}\Big|_{\mathscr{I}}$

<u>proof</u> (follow up)

We proved that $\mathcal{L}_n g_{\mu\nu}\big|_{\mathscr{I}} = \nabla_\mu n_\nu\big|_{\mathscr{I}} = \left(\nabla_e n^e\right)\frac{1}{4} g_{\mu\nu}$

This is an equation $\left(\mathcal{L}_n g_{\mu\nu}\right) dx^\mu dx^\nu$ in terms of the

4d coordinates $x^\mu = (\Omega, x^a)$

restricting to $\mathscr{I} \Leftrightarrow \Omega = 0$ this becomes

$$\mathcal{L}_n h_{ab} \propto h_{ab} \qquad \square$$

Proposition

changing the representatives $(\Omega, g_{\mu\nu}) \mapsto (\lambda(x)\Omega, \lambda^2(x)\Omega)$

changes the normal and metric at $\mathscr{I}$ as

$$(n^a, h_{ab}) \mapsto (\lambda_0(x^a)^{-1} n^a, \lambda_0^2(x) h_{ab}) \qquad \lambda_0(x^a) = \lambda(x^\mu)\big|_{\mathscr{I}}$$

proof: exercise

Definition (Geroch 77)

The previous proposition suggests to define the "universal structure of null infinity" as:

$$(n^a, h_{ab}) \sim (\lambda_0(x^a)^{-1} n^a, \lambda_0^2(x^a) h_{ab})$$

with $n^a h_{ab} = 0$, $\mathcal{L}_n h_{ab} \propto h_{ab}$

NB This can also be thought as a "conformal Carrollian" geometry see Duval - Gibbons - Horvathy - Zhang (2014)

Adapted coordinate system

This structure is "universal" for the following reason:

Proposition

One can always choose a coordinate system $(u, z, \bar{z})$ on $\mathscr{I}$

and a weyl rescaling $(n^a, h_{ab}) \mapsto (\lambda_0(x^a)^{-1} n^a, \lambda_0^2(x^a) h_{ab})$

such that

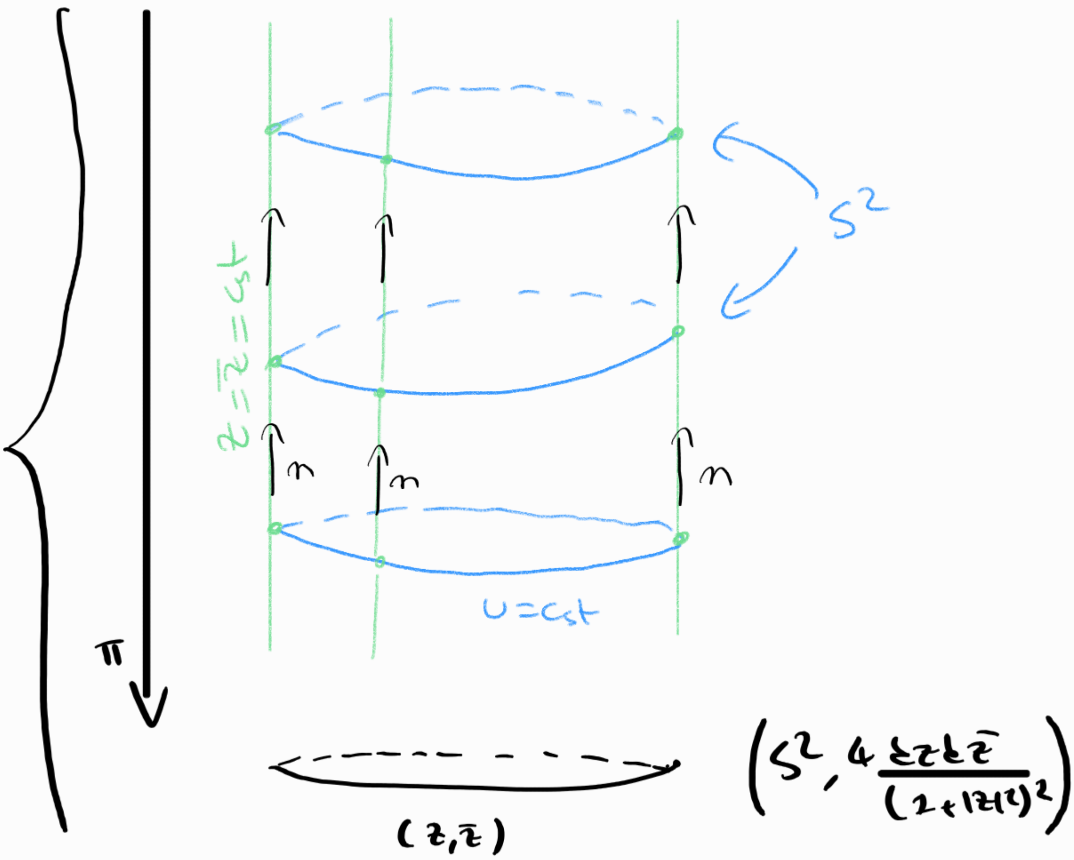
$$n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u} \qquad \text{and} \qquad h_{ab}\, dx^a dx^b = 0\,(du)^2 + \frac{4\, dz d\bar{z}}{(1+|z|^2)^2}$$

proof

see below

Fibre bundle

$\mathscr{I} \xrightarrow{\pi} S^2$



Extra

proof

- One can always choose a coordinate system $(\hat{u}, \hat{y}^A)$ such that $n = \hat{n}^u \frac{\partial}{\partial \hat{u}}$

$$\Rightarrow \quad h = 0\,(du)^2 + \hat{h}(u, y^A)_{AB}\, d\hat{y}^A d\hat{y}^B$$

- $\mathcal{L}_n h_{ab} \propto h_{ab}$ then implies $\hat{h}_{AB}(u, y^A) = f(u, y^A)\, h(y^A)_{AB}$

- Making use of the Weyl rescaling $(n^a, h_{ab}) \mapsto (\lambda_0(x^a)^{-1} n^a, \lambda_0^2(x^a) h_{ab})$ we can take $\hat{h}_{AB}(\hat{u}, \hat{y}^A) = h(\hat{y}^A)_{AB}$ } the $\hat{u}$ dependence has disappeared

- It is a classical results that any 2D metric is conformal to the round sphere so we can always find coordinates $\hat{y}^A \mapsto y^A = (z, \bar{z})$ such that

$$h_{ab}\, dy^a dy^b = f(z, \bar{z})\, \frac{4\, dz d\bar{z}}{(1 + |z|^2)^2}$$

- Making use of the Weyl rescaling $(n^a, h_{ab}) \mapsto (\lambda_0(x^a)^{-1} n^a, \lambda_0^2(x^a) h_{ab})$ we can take $\hat{h}(u, y^A) = \frac{4\, dz d\bar{z}}{(1 + |z|^2)^2}$ (none of the previous points changed the property $n^a \partial_a \propto \partial_u$)

- Finally we can always make a change of coordinate $\hat{u} \mapsto u(\hat{u}, \hat{y})$ such that

$$n = \frac{\partial}{\partial u}$$

The BMS group

We now work in the adapted coordinates of the previous proposition.

Proposition (Geroch 77)

The Lie algebra of infinitesimal symmetries of the universal structure,

i.e. the vector field X^a on $\mathscr{I}$ satisfying

$$\mathcal{L}_X h_{ab} = 2\alpha h_{ab} \quad , \quad \mathcal{L}_X n^a = -\alpha n^a$$

(for some function α on $\mathscr{I}$)

is generated by

$$\boxed{X = \left[\xi(z,\bar{z}) + \frac{u}{2}\left(\partial_z \chi + \partial_{\bar{z}}\bar{\chi}\right)\right]\frac{\partial}{\partial u} + \chi(z)\frac{\partial}{\partial z} + \bar{\chi}(\bar{z})\frac{\partial}{\partial \bar{z}}}$$

where $\xi(z,\bar{z})$ is a function on S^2

$\chi(z)\frac{\partial}{\partial z} = (a + bz + cz^2)\frac{\partial}{\partial z}$ is a generator of Möbius transformations $PSL(2,\mathbb{C}) \simeq SO(3,1)$

$$\boxed{\begin{array}{l}\text{This Lie algebra is isomorphic to} \\ \qquad Lie(so(3,1)) \ltimes \mathcal{C}^\infty(S^2)\end{array}}$$

BMS Lie algebra

⚠ In this semi direct product, the action of $so(3,1)$ on $\mathcal{C}^\infty(S^2)$ acts as on conformal primary of weight $\Delta = -1$.

<u>proof</u>

- One obtains the expression for the generator by a direct coordinate computation.

- We note $X_\xi = \xi(z,\bar z)\,\partial_u$

$$X_\chi = u\,\frac{1}{2}\left(\partial_z \chi + \partial_{\bar z}\bar\chi\right)\partial_u + \chi\,\partial_z + \bar\chi\,\partial_{\bar z}$$

The algebra is:

i) $[X_{\xi_1}, X_{\xi_2}] = 0$

$\Longrightarrow$ thus the generators X_ξ form an ∞ dimensional abelian Lie algebra isomorphic to $\mathcal{C}^\infty(S^2)$

ii) $[X_{\chi_1}, X_{\chi_2}] = X_{(\chi_1 \partial_z \chi_2 - \chi_2 \partial_z \chi_1)}$

$\Longrightarrow$ the generators X_χ form an $so(3,1)$ Lie algebra

iii) $[X_\chi, X_\xi] = \left(\left(\chi\partial_z + \bar\chi\partial_{\bar z}\right)\xi - \frac{1}{2}\left(\partial_z\chi + \partial_{\bar z}\bar\chi\right)\xi\right)\partial_u$

$\Longrightarrow$ This is the action of $so(3,1)$ on a conformal primary of conformal weight -1.

Penrose peeling

Proposition

The Weyl tensor is conformally invariant:

if $\hat{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$, $\quad \widetilde{W}^{\mu}{}_{\nu\rho\sigma} = W^{\mu}{}_{\nu\rho\sigma}$.

proof

This is a classical result which can be proved by direct computation.

Theorem (Penrose 1963)

If the unphysical metric $g_{\mu\nu}$ is at least $\mathcal{C}^4$ at $\mathscr{I}$, then

$$\left.\widetilde{W}^{\mu}{}_{\nu\rho\sigma}\right|_{\mathscr{I}} = \left.W^{\mu}{}_{\nu\rho\sigma}\right|_{\mathscr{I}} = 0.$$

This is Penrose's peeling

proof

see the paper (see also next lecture)

Remark

We can define NP coefficients with the same expressions as in lecture 1

but now this is n^{μ} which is kept fixed in the definition of an adapted

null tetrad $(\ell^{\mu}, m^{\mu}, n^{\mu})$ (⚠ this is a null tetrad for the unphysical metric)

and the invariant component is:

$$\boxed{\Psi_4^{\circ} := r\, W_{\mu\nu\rho\sigma} n^{\mu} \bar{m}^{\nu} n^{\rho} \bar{m}^{\sigma}\Big|_{\mathscr{I}}}$$

"Gravitational field"

and then $\Psi_3^{\circ}, \Psi_2^{\circ}, \Psi_1^{\circ}, \Psi_0^{\circ}$ in that order.

Proposition

Penrose peeling implies Sachs peeling along any hypersurface orthogonal null geodesic congruence reaching $\mathscr{I}$.

Extra

proof

Consider a null tetrad $(\tilde{e}^\mu, \tilde{m}^\mu, \tilde{\bar{m}}^\mu, n^\mu)$ for $\hat{g}_{\mu\nu}$

⚠ here n^μ is the normal at $\mathscr{I}$: $n^\mu|_{\mathscr{I}}$ is finite

since $\hat{g}_{\mu\nu} = r^2 g_{\mu\nu}$ and $\tilde{e}^\mu n^\nu \hat{g}_{\mu\nu} = -1$

there must exist e^μ s.t $e^\mu|_{\mathscr{I}}$ is finite and $\tilde{e}^\mu = \frac{1}{r^2} e^\mu$

similarly since $\tilde{m}^\mu \tilde{\bar{m}}^\nu \hat{g}_{\mu\nu} = 1$

there must exist m^μ s.t. $m^\mu|_{\mathscr{I}}$ is finite and $\tilde{m}^\mu = \frac{1}{r} m^\mu$

NB: then $(e^\mu, m^\mu, \bar{m}^\mu, n^\mu)$ restricts to a null tetrad at $\mathscr{I}$

finally since $\tilde{W}_{\mu\nu\rho\sigma} = \hat{g}_{\mu\alpha} \tilde{W}^\alpha{}_{\nu\rho\sigma} = r^2 W_{\alpha\nu\rho\sigma} = r^2 \left(O(r^{-1})\right) = O(r)$

then $\Psi_4 = \underbrace{\tilde{W}_{\mu\nu\rho\sigma}}_{O(r)} n^\mu \underbrace{m^\nu}_{O(r^{-1})} n^\rho \underbrace{m^\sigma}_{O(r^{-1})} = r^{-1} \Psi_4^0 + O(r^{-2})$

$\vdots$

$\Psi_0 = \underbrace{\tilde{W}_{\mu\nu\rho\sigma}}_{O(r)} \underbrace{e^\mu}_{O(r^{-2})} \underbrace{m^\nu}_{O(r^{-1})} \underbrace{e^\rho}_{O(r^{-2})} \underbrace{m^\sigma}_{O(r^{-1})} = r^{-5} \Psi_0^0 + O(r^{-6})$

Sachs' Peeling

□

Remarks

- It is remarquable that Penrose's simple definition of asymptotically flat spacetime implies Sach's peeling.

- The BMS group here immediately appears as the group of symmetries of the universal/carrollian structure at $\mathscr{I}$.

- In the following lecture, we will see that asymptotic flatness is equivalent to the existence of BMS coordinates.

  The BMS group will then be realised as asymptotic symmetries.

- The peeling must hold is $g_{\mu\nu}$ is differentiable enough along $\mathscr{I}$.

$\Rightarrow$ Nowaday the presence/lack of peeling is usually thought as a smoking gun for presence/lack of differentiability (and not as a fundamentally relevant physical notion).