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Lecture 4:
Asymptotically flat spacetime (2)
BMS coordinates etc

Goal of these last two lectures:

(1) Investigate some consequences of Penrose's definition for "asymptotically flat spacetimes"

(2) Show the equivalence with BMS coordinates and relate to the first lecture

BMS coordinates

Let $(\hat{M}, \hat{g}_{\mu\nu})$ be an asymptotically flat spacetime.
We already saw that one can always find adapted coordinates $(u, y^A = (z, \bar{z}))$ such that

$$n^a \frac{\partial}{\partial x^a} = \partial_u, \qquad h_{ab}\, dx^a dx^b = \frac{4\, dz d\bar{z}}{(1+|z|^2)^2} = h^{(S^2)}_{ab}\, dx^a dx^b$$

Proposition

Once we have made a choice of coordinate system $(u, z, \bar{z})$ on $\mathscr{I}$, there is a unique coordinate extension $(u, r = \Omega^{-1}, z, \bar{z})$ in a neighborhood of $\mathscr{I}$ such that

$$ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$$

$$= \frac{1}{\Omega^2}\left(\Omega^3 e^{2\beta} V (du)^2 + e^{2\beta}\, 2 du\, d\Omega + H_{AB}\left(dy^A - U^A du\right)\left(dy^B - U^B du\right)\right)$$

$$= \frac{e^{\beta} V}{r}(du)^2 - e^{2\beta}\, 2 du dr + r^2 H_{AB}\left(dy^A - U^A du\right)\left(dy^B - U^B du\right)$$

with $r^{-3} V(u, r, y^A) = o(1)$ $\qquad U^A(u, r, y^A) = o(1)$

$\beta(u, r, y^A) = o(1)$ $\qquad H_{AB}(u, r, y^A) = h^{(S^2)}_{AB}(z, \bar{z}) + o(1)$

sphere metric of radius 1

and

$\partial_r \det H = 0$

BMS gauge

Some remarks

- at $\Omega = 0$ these coordinates coincides with the adapted coordinates at $\mathscr{I}$

- comparing with lecture 1 one sees that these define a null geodesic congruence starting at $\mathscr{I}$.

- here the condition $\partial_r \left(\det(H_{AB})\right) = 0$ distinguish the BMS gauge from the Newman-Unti gauge from the first lecture.

proof

- the adapted coordinates (u, y^A) at $\mathscr{I}$ defines a frame at $\mathscr{I}$:

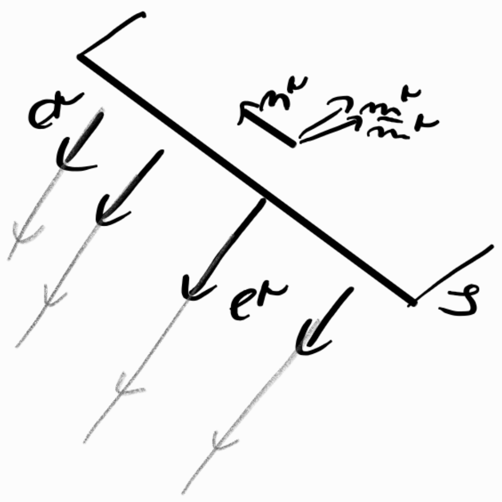
$$\left(n\Big|_{\mathscr{I}} = \frac{\partial}{\partial u} \;,\; m\Big|_{\mathscr{I}} = \frac{(1+|z|^2)}{\sqrt{2}}\frac{\partial}{\partial z} \;,\; \bar{m}\Big|_{\mathscr{I}} = \frac{(1+|z|^2)}{\sqrt{2}}\frac{\partial}{\partial \bar{z}} \right)$$

which can be uniquely completed into a null tetrad at $\mathscr{I}$:

$$\left(n|_{\mathscr{I}} = \frac{\partial}{\partial u} ,\; m|_{\mathscr{I}} \;,\; \bar{m}|_{\mathscr{I}} \;,\; e \right)$$

" $e = \frac{\partial}{\partial r}$ "

- by construction e^μ is a null vector transverse to $\mathscr{I}$:



We can thus consider the null geodesic congruence it generates

= solve the geodesic equation for $g_{\mu\nu}$

Since null geodesic are conformal invariants, this also solves the geodesic equation for $\tilde{g}_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}$

$\Longrightarrow$ this gives a null geodesic congruence (for $g_{\mu\nu}$) ad starting at $\mathscr{I}$

- We can now extend (u, y^A) be requiring that they are constant along the null geodesic congruence.

$\Rightarrow$ together with Ω, this defines a coordinate system (u, y^A, Ω) such that:

$$g_{\mu\nu} dx^\mu dx^\nu = \Omega^3 e^{2\beta} V (du)^2 + e^{2\beta} 2 du d\Omega + H_{AB}\left(dy^A - U^A du\right)\left(dy^B - U^B du\right)$$

because:

i) $g_{\Omega\Omega} = 0$ because $\ell = \frac{\partial}{\partial \Omega}$ generates the null geodesic congruence

ii) $\partial_r g_{A\Omega} = 0$ because ℓ is geodesic $\nabla_\ell \ell^\nu \propto \ell^\mu$

iii) $g_{\tau\Omega}\big|_S = g(\partial_\tau, \partial_\Omega) = m^\mu \ell^\nu g_{\mu\nu}\big|_S = 0$ by definition of ℓ^μ

- Since $g_{\mu\nu}\big|_S$ is finite we must have

$$r^{-3} V(u,r,y^A) = o(1) \qquad U^A(u,r,y^A) = o(1)$$

$$\beta(u,r,y^A) = o(1) \qquad H_{AB}(u,r,y^A) = h^{(S^2)}_{AB}(z,\bar z) + o(1)$$

sphere metric of radius 1

- Finally the ambiguity in Ω : $\Omega \mapsto f(x)\,\Omega$

allows to fix $\quad \partial_\Omega \det H = 0 \quad \}$ BMS gauge

(we could choose instead $\beta = 0 \quad \}$ Newman-Unti gauge)

Theorem BMS (1962)

Assuming that the coefficients in the above metric admit an expansion of the form:

$$r^{-3}V(u,r,y^A) = \sum_{i=0}^{\infty} V_{(i)}(u,y^A)\, r^{-i}, \quad U^A(u,r,y^A) = \sum_{i=0}^{\infty} U^A_{(i)}(u,y^A)\, r^{-i},$$

$$\beta(u,r,y^A) = \sum_{i=0}^{\infty} \beta_{(i)}(u,y^A)\, r^{-i}, \quad H_{AB}(u,r,y^A) = \sum_{i=0}^{\infty} H^{(i)}_{AB}(u,y^A)\, r^{-i}.$$

One can solve Einstein's equation order by order in r.
As a result, one finds:

2-sphere metric "Asymptotic shear"

$$H_{AB} = \boxed{h^{(S^2)}{}_{AB}} + r^{-1}\boxed{C_{AB}(u,z,\bar{z})} + \underbrace{r^{-2}D_{AB}(u,z,\bar{z}) + r^{-3}E_{AB} + \dots}$$

$\Rightarrow$ it is symmetric trace-free but otherwise unconstrained

at each order these fields are constrained by evolution equations: e.g. $\partial_u D_{AB} = 0$, $\partial_u E_{AB} = fct(C_{AB})$ etc.

$$\beta = 0 + r^{-2}\left(-\frac{1}{64}C^{AB}C_{AB}\right) + \dots$$

} at all orders, all terms are fixed by Einstein's equation.

$$r^{-3}V = 0 - r^{-2}\left(\frac{1}{2}\right) + r^{-3}\, 2\boxed{M(u,z,\bar{z})} + \dots$$

The "Mass aspect" is constrained by an evolution equation $\partial_u M = \dots$, subleading terms are completely fixed.

$$U^A = 0 - r^{-2}\left(\frac{1}{2}\nabla_C C^C{}_A\right) + r^{-3}\boxed{N^A(u,z,\bar{z})} + \dots$$

The "angular momentum aspect" is constrained by an evolution equation $\partial_u N^A = \dots$, subleading terms are completely fixed.

Some remarks

- The shear of this null geodesic congruence is

$$\sigma_{AB} = \partial_r H_{AB} = r \underbrace{C_{AB}}_{} + o(r)$$

hence the name asymptotic shear.

- The Bondi News is defined as $N_{AB} := \partial_u C_{AB}$

it sources the evolution equation for the mass aspect:

$$\boxed{\underbrace{\partial_u \int_{S^2} \Pi}_{\text{Variation of energy in time}} = \underbrace{-\frac{1}{2}\int_{S^2} N^{AB} N_{AB}}_{\text{Flux term, it is negative}}}$$

Bondi mass loss formula

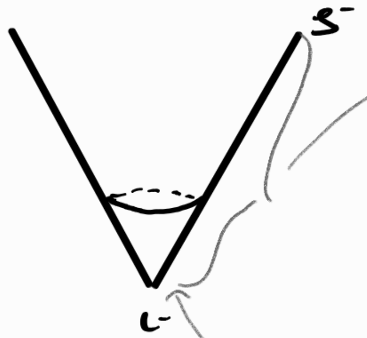
Flux term it is negative = energy is flowing away from the spacetime.

- The NP coefficient Ψ_4 behaves as

$$\boxed{\Psi_4 = r^{-1} \partial_u N_{\bar{z}\bar{z}} + \mathcal{O}(r^{-2})}$$

"Radiation fields"

- The free data in this expansion are



- A free function along $\mathscr{I}^-$

$$\boxed{N_{AB}(u,y^A) := \partial_u C_{AB}(u,y^A)}$$

"News tensor"
= gravitational radiations

- An infinite tower of zero modes at i^-

$$C_{AB}\big|_{v=-\infty}, \; \Pi\big|_{v=-\infty}, \; \hat{N}\big|_{v=-\infty}, \; D_{AB}\big|_{v=-\infty},$$
$$E_{AB}\big|_{v=-\infty}, \; \dots$$

This is typical of a characteristic problem = solving PDE from data on a null surface (as opposed to usual initial value problem from space-like hypersurface.

$\Longrightarrow$ In particular "constraints" appear here as evolution equation on the zero modes but the News is a free function.

- We found, as a result of imposing the full non linear Einstein's equations:

$$ds^2 = \frac{1}{\Omega^2}\Big[\, 2\,du\,d\Omega + h^{(S^2)}_{AB}\,dy^A dy^B + \Omega\big(C_{AB}\,dy^A dy^B\big) + O(\Omega^2)\,\Big]$$

- Linearizing this result over Minkowski space, we find:

$$ds^2 = \big(\eta_{\mu\nu} + \delta g_{\mu\nu}\big)\,dx^\mu dx^\nu$$

$$= \frac{1}{\Omega^2}\Big[\,\Omega^2 (du)^2 + 2\,du\,d\Omega + h^{(S^2)}_{AB}\,dy^A dy^B\Big] + \Big[\frac{1}{\Omega}\,\delta C_{AB}\,dy^A dy^B + O(\Omega^{-2})\Big]$$

hence one cannot tell from the asymptotics if the asymptotic shear is the initial value for the full or the linearized equations.

$\Longrightarrow$ This is the foundational basis for a scattering problem.

Asymptotic symmetries

Theorem (BMS 1962)

The space of vector fields

$$\xi = \xi^u \partial_u + \xi^z \partial_z + \xi^{\bar z} \partial_{\bar z} + \xi^r \partial_r$$

which generates an infinitesimal symmetry of the previous expansion

i.e. such that $\delta g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$ does not change the form of the expansion.

is of the form:

$$\xi^u = \boxed{f(z,\bar z)} + \frac{u}{2}\left(\nabla_z Y^z + \nabla_{\bar z} \bar Y^{\bar z}\right) + O(r^{-1})$$

$$\xi^z = \boxed{Y^z(z)} + O(r^{-1})$$

$$\xi^{\bar z} = \boxed{\bar Y^{\bar z}(\bar z)} + O(r^{-1})$$

$$\xi^r = r\left(\nabla^2 f - \frac{u}{2}\left(\nabla_z Y^z + \nabla_{\bar z} \bar Y^{\bar z}\right) + O(r^{-1})\right)$$

free data

Subleading orders are fixed in terms of $\left(f(z,\bar z), Y(z), \bar Y(\bar z)\right)$

These form an infinite dimensional Lie algebra isomorphic to

$$\boxed{Lie\left(so(3,1)\right) \ltimes C^\infty(S^2)}$$

BMS Lie algebra

Remarks

- The restriction of the asymptotic symmetry generators to $\mathscr{I}$ gives the generator of symmetries for the universal / Carrollian structure:

$$X = \left(f(z,\bar{z}) + \frac{u}{2}\left(\nabla_z Y^z + \nabla_{\bar{z}} \bar{Y}^{\bar{z}} \right) \right) \frac{\partial}{\partial u} + Y \partial_z + \bar{Y} \partial_{\bar{z}}$$

- Since the initial data for the scattering live at $\mathscr{I}$ it is clear that the BMS group must act on them.

⚠ These are not tensors on $\mathscr{I}$ ⚠