Sabrina @ Warsaw

Elements of Celestial Holography

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Lecture 1: Overview + IR triangle primer
Lecture 2: Asymptotic Symmetries & Soft theorems
Lecture 3: Kinematics of scattering → Celestial Amplitudes
Lecture 4: The Celestial Dictionary

Lecture 1

Why? Quantum Gravity #goals

Quantum Gravity $\xrightarrow{\text{consistency in UV}}$ String Theory

String Theory $\xrightarrow{\text{AdS/CFT correspondence}}$ Holography

BH thermo. → Holography

A theory of quantum gravity can be encoded in a lower dimensional quantum theory without gravity @ the spacetime boundary

Celestial Holography: apply holo. princ. to $\Lambda = 0$ spacetimes

can start w/ a bottom up approach that matches symmetries!

Yannick: boundary & symmetries

Me: symmetries → scattering → celestial CFT (CCFT)

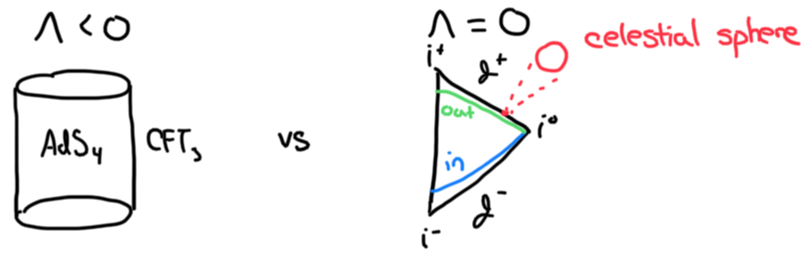
Tomasz: scattering

Piotr: CFT

+ top down

Tim: twistors

Big Picture:



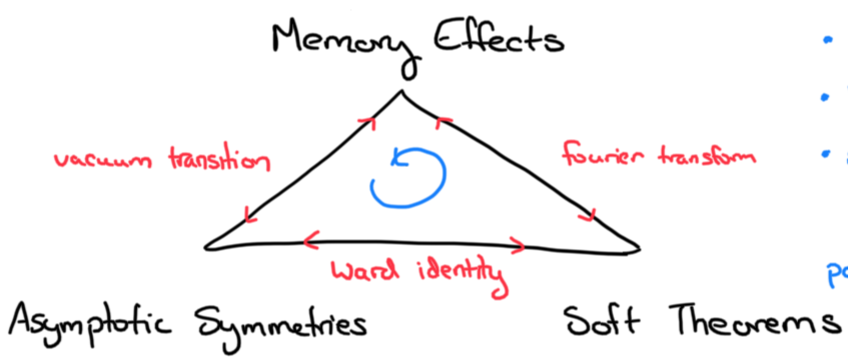
flat is different ... but not ~~scarily~~ fun! so

- causal structure of boundary is different → Carrollian CFTs / → dim. red. to Celestial CFT
  - ↓ symmetry enhancements
- can still view scattering as bndy data, prep. $|in\rangle$ & $\langle out|$ w/ operators at $\mathscr{I}^\pm$, $i^\pm$

Themes:

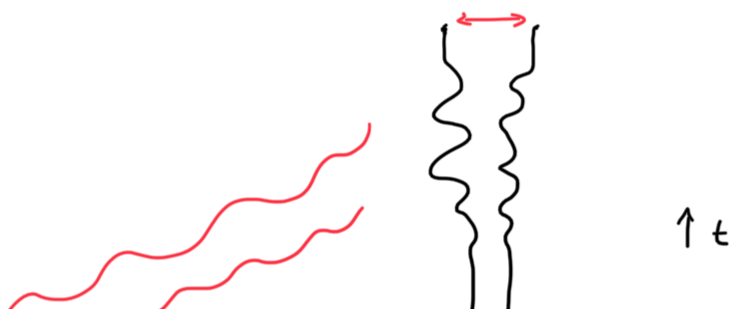
- intrinsic desc. @ bndy inspired by pull back from bulk
- various changes of bases

The IR Triangle



- leading E&M
- leading grav.
- subleading grav.
- ...

pattern → predictions



$$\partial_\tau^2 s^\mu = R^\mu{}_{\lambda\rho\nu} t^\lambda t^\rho s^\nu$$
$$\downarrow$$
$$\partial_u^2 s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \partial_u^2 C_{zz} s^z$$
$$\downarrow$$
$$\Delta s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \Delta C_{zz} s^z$$

$u = t - r$, $z = e^{i\phi}\tan\frac{\theta}{2}$

see Yannick lects.

$t^\lambda \partial_\lambda = \partial_u$, $\tau \sim u$, $R_{zuzu} \sim -\frac{1}{2} r \partial_u^2 C_{zz}$

Exercise 13 of Strominger lect.

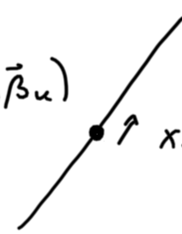
$\exists$ non-trivial tail behavior of grav waveform

- $\rightarrow$ meas. w/ asymp. detectors — Mem. Eff.
- $\rightarrow$ $\Theta(u) \xrightarrow{\text{F.T.}} \frac{1}{\omega}$ — soft pole — Taylor
- $\rightarrow$ $\Delta C_{zz} = -2 D_z^2 \Delta C$ — vac. trans. — Yannick

A U(1) Example

$$S = -\frac{1}{4e^2}\int d^4x \sqrt{-g}\, F_{\mu\nu}F^{\mu\nu} + S_M \xRightarrow{\frac{\delta S}{\delta A_\nu}} \nabla^\mu F_{\mu\nu} = e^2 j_\nu^M$$

$$j_\mu^M(x) = \sum_{k=1}^n Q_k \int d\tau\, U_{k\mu}\, \delta^4(x^\nu - U_k^\nu \tau) \qquad U_k^\mu = \gamma_k(1, \vec\beta_k)$$



$x_k^\mu(\tau) = U_k^\mu \tau$

$$F_{rt}(\vec x, t) = \frac{e^2}{4\pi}\sum_{k=1}^n \frac{Q_k \gamma_k (r - t\hat x\cdot\vec\beta_k)}{|\gamma_k^2(t - r\hat x\cdot\vec\beta_k)^2 - t^2 + r^2|^{3/2}}$$

$F_{rt} = F_{ru}$ since anti-sym & $u = t - r$

$r \to \infty$ u fixed $\longrightarrow \mathscr{I}^+$

$$F_{ru}\big|_{\mathscr{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^n \frac{Q_k}{\gamma_k^2(1 - \hat x\cdot\vec\beta_k)^2}$$

$r \to \infty$ $v = t + r$ fixed $\longrightarrow \mathscr{I}^-$

$$F_{rv}\big|_{\mathscr{I}^-} = \frac{e^2}{4\pi r^2}\sum \frac{Q_k}{\gamma_k^2(1 + \hat x\cdot\vec\beta_k)^2}$$

$$\left.\lim_{r\to\infty} r^2 F_{ru}(\hat x)\right|_{\mathscr{I}^+_-} = \left.\lim_{r\to\infty} r^2 F_{rv}(-\hat x)\right|_{\mathscr{I}^-_+}$$

$$Q_E = \frac{1}{e^2}\int_{S^2} \star F = \lim_{r\to\infty} \frac{1}{e^2}\int_{S^2} d^2z\sqrt{\gamma}\, r^2 F_{ru}$$

$$\overset{?}{\longrightarrow} Q(\lambda) = \frac{1}{e^2}\int_{S^2} \lambda(z,\bar{z}) \star F$$

take-aways
- boost gives non-triv angle dep.
- anti-podal matching

Next time
deriv. of soft thm = Ward Id. for $U(1)$

Lecture 2

Goal: Demonstrate Asymptotic Sym. $\Leftrightarrow$ Soft theorem for $U(1)$ example

Last time we saw that for L.W. sol'n

$$Q^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon \star F = \frac{1}{e^2}\int_{\mathcal{I}^-_+} \varepsilon \star F = Q^-_\varepsilon$$

for $\varepsilon(z,\bar{z})\big|_{\mathcal{I}^+_-} = \varepsilon(z,\bar{z})\big|_{\mathcal{I}^-_+}$ since $F^{(2)}_{ru}(z,\bar{z})\big|_{\mathcal{I}^+_-} = F^{(2)}_{rv}(z,\bar{z})\big|_{\mathcal{I}^-_+}$

What is Q_ε? 1407.3789 & 1901.01622

$\nabla^\nu F_{\nu\mu} = e^2 j^M_\mu$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\nabla^\mu j^M_\mu = 0$

$\exists$ gauge sym $\delta_\varepsilon A_\mu = \partial_\mu \varepsilon$ Let's gauge fix!

$\nabla^\mu A_\mu = 0$ still allows ε s.t. $\Box\varepsilon = 0$

Notation Aside
$\mathcal{O}(u,r,z,\bar{z}) = \sum_n r^{-n}\mathcal{O}^{(n)}(u,z,\bar{z})$

free data $\varepsilon^{(1)}(u,z,\bar{z}) \longrightarrow A^{(1)}_u = 0$

still can have $\varepsilon^{(0)}(z,\bar{z})$ Q_ε generates this transformation

leading r-behavior of u-component of eom:

$$\partial_u F^{(2)}_{ru} + D^z F^{(0)}_{uz} + D^{\bar z} F^{(0)}_{u\bar z} + e^2 j^{(2)}_u = 0$$

$$\downarrow$$

$$Q^+_\varepsilon = \underbrace{-\frac{1}{e^2}\int_{\mathcal{I}^+} du\, d^2z \left(\partial_z \varepsilon F^{(0)}_{u\bar z} + \partial_{\bar z}\varepsilon F^{(0)}_{uz}\right)}_{Q^+_S} + \underbrace{\int_{\mathcal{I}^+} du\, d^2z\, \varepsilon \gamma_{z\bar z} j^{(2)}_u}_{Q^+_H}$$

indeed $[Q^+_\varepsilon, \Phi^M(u,z,\bar z)] = [Q^+_H, \Phi^M(u,z,\bar z)] = -q\varepsilon(z,\bar z)\Phi^M(u,z,\bar z)$

* some care must be taken re: the brackets on the zero-modes of the radiative phase space (HW: read 2.6)

we would like to show

$$\langle \text{out}|\, Q^+_\varepsilon S - S Q^-_\varepsilon \,|\text{in}\rangle = 0$$

we know $Q^\pm_H$ measures the matter charges @ $\mathcal{I}^\pm$. Let's evaluate Q_S!

$$A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon^{\alpha *}_\mu(\vec q)\, a_\alpha(\vec q) e^{iq\cdot x} + \varepsilon^\alpha_\mu(\vec q)\, a_\alpha(\vec q)^\dagger e^{-iq\cdot x}\right]$$

for our gauge $F^{(0)}_{uz} = \partial_u A^{(0)}_z - \cancel{\partial_z A^{(0)}_u}^{\to 0}$

$$A^{(0)}_z = \lim_{r\to\infty} \partial_z x^\mu A_\mu(x)$$

now at large r fixed u

$$e^{iq\cdot x} = e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} \longrightarrow e^{-i\omega_q u} \times \frac{1}{\omega_q r}\frac{\delta(\theta)}{\sin\theta}$$

$$A^{(0)}_z = \frac{-i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar z}\int_0^\infty d\omega_q \left[a_+(\omega_q \hat x) e^{-i\omega_q u} - a_-(\omega_q\hat x)^\dagger e^{i\omega_q u}\right]$$

$$\int du\, F^{(0)}_{uz} = \frac{-1}{8\pi}\frac{\sqrt{2}e}{1+z\bar z}\lim_{\omega\to 0^+}\left[\omega a_+(\omega\hat x) + \omega a_-(\omega\hat x)^\dagger\right]$$

but Weinberg tells us that these insertions have a universal form!

$$\langle out | a_+(\vec{q}) S | in \rangle = e \sum_{out-in} \frac{Q_k p_k \cdot \varepsilon^+}{p_k \cdot q} \langle out | S | in \rangle + \mathcal{O}(\omega_q^0)$$

Plugging in the soft thm.

$$\langle out | \int du F_{uz}^{(0)} S | in \rangle = -\frac{e^2}{4\pi} \sum_{out-in} \frac{Q_k}{z - z_k} \langle out | S | in \rangle$$

and from evaluating

$$\langle out | Q^+_{S,\varepsilon} S - S Q^-_{S,\varepsilon} | in \rangle = -\langle out | Q^+_{H,\varepsilon} S - S Q^-_{H,\varepsilon} | in \rangle$$

we indeed have

$$\langle out | Q^+_\varepsilon S - S Q^-_\varepsilon | in \rangle = 0$$

But look! $j^+ := Q^+_S\left(\varepsilon = \frac{1}{z-w}\right) = -4\pi \int du F_{uz}$ obeys

$$\langle j(z) \mathcal{O}_1(z_1, \bar{z}_1) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle = \sum_k \frac{Q_k}{z - z_k} \langle \mathcal{O}_1(z_1, \bar{z}_1) \ldots \mathcal{O}_n(z_n, \bar{z}_n) \rangle$$

ASG $\Rightarrow$ 2D KM sym of S-matrix?

Takeaways
- soft thm = ASG Ward Id
- ∞-dim angle dep enhancement
- look like 2D currents

Next time
- reorganize scattering to make these symmetries manifest!

Lecture 3

Last time: soft thm = Ward Id $\Rightarrow$ 2D current

Focused on U(1) ex. but from lect. 1 know it should generalize

Sub soft grav. $\Rightarrow$ 2D Stress tensor

$$T_{zz} = -i\frac{3!}{8\pi G}\int d^2w \frac{1}{(z-w)^4}\int du\, u \partial_u C^{w}{}_{\bar{w}}$$

$$\langle T_{zz}\mathcal{O}_1\ldots\mathcal{O}_n\rangle = \sum_k \left[\frac{h_k}{(z-z_k)^2} + \frac{\partial_{z_k}}{z-z_k}\right]\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle$$

$h_k = \frac{1}{2}(s_k - \omega_k\partial_{\omega_k})$ diagonalized for boost eigenstates

Today: Kinematics of Scattering $\longrightarrow$ Celestial Amplitudes

Claim: ASG sym enhancements naturally organized in terms of a CFT_2

$SL(2,\mathbb{C}) \simeq$ Lorentz $\subset$ Poincaré Global Conf.

+ gravity

$\downarrow$

Vir $\times$ Vir $\simeq$ Superrotations $\subset$ BMS Larger Sym. Multiplets!

$${}_{\text{boost}}\langle \text{out}|S|\text{in}\rangle_{\text{boost}} = \langle \mathcal{O}^{\pm}_{\Delta_1,J_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{\pm}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle_{CCFT}$$

is just a change of basis!

Let's start with the global part. We can prepare scattering states that are 2D primaries with an appropriate choice of wave packet.

Def'n: A conformal primary wavefunction is a function on $\mathbb{R}^{1,3}$ which transforms under $SL(2,\mathbb{C})$ as

$$\Phi^{s}_{\Delta,J}\left(\Lambda^{\mu}{}_{\nu}X^{\nu};\frac{aw+b}{cw+d},\frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right) = (cw+d)^{\Delta+J}(\bar{c}\bar{w}+\bar{d})^{\Delta-J}D_s(\Lambda)\Phi^{s}_{\Delta,J}(X^{\mu};w,\bar{w})$$

for on-shell states we also impose the spin-s linearized bulk eoms.

Then lorentz inv. of the innerproduct guarantees

$$\mathcal{O}^{s,\pm}_{\Delta,J}(w,\bar{w}) = i\left(\hat{\mathcal{O}}^s(X), \Phi^s_{\Delta,-J}(X^\mu_\mp; w,\bar{w})\right)_\Sigma$$

$X^0_\pm = X^0 \mp i\varepsilon$

is a 2D primary operator.

Easy to construct. Using

$$q^\mu = (1+w\bar{w}, w+\bar{w}, i(\bar{w}-w), 1-w\bar{w}) \qquad \epsilon^\mu_w = \frac{1}{\sqrt{2}}\partial_w q^\mu$$

we can construct a null tetrad

$$l^\mu = \frac{q^\mu}{-q\cdot X} \quad n^\mu = X^\mu + \frac{X^2}{2} l^\mu \quad m^\mu = \varepsilon^\mu_w + (\epsilon_w\cdot X)\, l^\mu$$

and we have

$$\Phi^s_{\Delta,J=+s} = m_{\mu_1}\cdots m_{\mu_s}\frac{f(X^2)}{(q\cdot X)^\Delta}$$

This wavefn. vers. readily covers $m=0$ or $m\neq 0$ cases.

For $m=0$ things are simpler

$$A^\pm_{\mu;\Delta,J=+1} = m_\mu \frac{1}{(q\cdot X_\pm)^\Delta} = c(\Delta)\, \varepsilon_{w;\mu}\int_0^\infty dw\, w^{\Delta-1} e^{\pm i w q\cdot X - \varepsilon w} + \nabla_\mu \lambda^\pm_{\Delta,J}$$

can just Mellin transform the p^μ-eigenstates!

Let's look at

$$|\Delta,0,0;s\rangle = \int_0^\infty dw\, w^{\Delta-1}\, |p = w(1,0,0,1); s\rangle$$

Then $P^0 - P^3$, $P^1 \pm i P^2$ annihilate this state

while the combinations $(K_i = M_{i0} \qquad J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk})$

$$L_0=\tfrac{1}{2}(J_3-iK_3),\quad L_{-1}=\tfrac{1}{2}(-J_1+iJ_2+iK_1+K_2)\quad L_1=\tfrac{1}{2}(J_1+iJ_2-iK_1+K_2)$$

$$\bar{L}_0=\tfrac{1}{2}(-J_3-iK_3),\quad \bar{L}_{-1}=\tfrac{1}{2}(J_1+iJ_2+iK_1-K_2)\quad \bar{L}_1=\tfrac{1}{2}(-J_1+iJ_2-iK_1-K_2)$$

which obey two copies of the Witt algebra

$$[L_m,L_n]=(m-n)L_{n+m}\quad [\bar{L}_m,\bar{L}_n]=(m-n)\bar{L}_{n+m}\quad [L_n,\bar{L}_m]=0$$

act as follows

$$L_0|\Delta,0,0;s\rangle=\tfrac{1}{2}(\Delta+s)|\Delta,0,0;s\rangle\quad L_1|\Delta,0,0;s\rangle=0$$

Some comments:

- $\omega\in(0,\infty)\rightarrow\Delta\in 1+i\lambda$ Principal series spectrum to capture radiative phase space
- $p^0+p^3:\ \Delta\rightarrow\Delta+1$ so some care should be taken
- Poincaré Primaries $\rightarrow$ BMS primaries (multiplets w/ soft gravitons)
- Can Mellin transform the amplitude directly

$$\langle\mathcal{O}^{\pm}_{\Delta_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{\pm}_{\Delta_n}(z_n,\bar{z}_n)\rangle=\prod_{i=1}^{n}\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\langle out|S|in\rangle$$

Will use this to fill in the dictionary next time!

Lecture 4

Last time: Saw that we can Mellin transform amplitudes to correlators in a 2D Celestial CFT.

$$\langle\mathcal{O}^{\pm}_{\Delta_1}(z_1,\bar{z}_1)\ldots\mathcal{O}^{\pm}_{\Delta_n}(z_n,\bar{z}_n)\rangle=\prod_{i=1}^{n}\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\langle out|S|in\rangle$$

Today: use knowledge of amplitudes to build the Celestial dictionary.

Outline: 1. Spectrum $\to$ soft limits as currents; $\to$ shortened multiplets

2. OPE $\to$ OPE from collinear splitting function; $\to$ $w_{1+\infty}$ symmetry algebra

If we had a 2D CFT we would be able to use the symmetries to build correlators, just given the CFT data.

Last time we saw that: $\cdot\ \Delta = 1 + i\lambda$ captures radiative states

$\cdot\ p^+$ moves us off the principal series

We want to look at the analytic structure for $\Delta \in \mathbb{C}$

$$A := \langle \text{out} | S | \text{in} \rangle \sim \omega^{-1} A^{(1)} + A^{(0)} + \dots$$

using $\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \omega^{\varepsilon - 1} = \delta(\omega)$ we have

$$\lim_{\Delta \to -n} (\Delta + n) \int_0^\infty d\omega\, \omega^{\Delta - 1} \sum_k \omega^k A^{(k)} = A^{(n)}$$

$\exists$ pole at -ive integer Δ whose residues are terms in the soft exp.

Now these modes have special $SL(2,\mathbb{C})$ multiplets

$$[L_1, (L_{-1})^k] = k (L_{-1})^{k-1} (2L_0 + k - 1)$$

$$\Downarrow$$

$$L_1 (L_{-1})^k |h, \bar h\rangle = k(2h + k - 1)(L_{-1})^{k-1} |h, \bar h\rangle$$

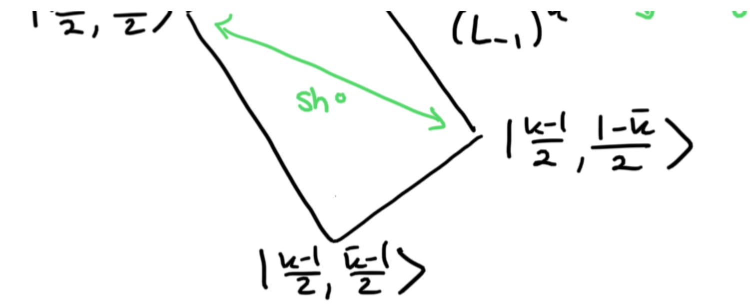
$\exists$ a primary descendant when $h = \frac{1-k}{2}$ for $k \in \mathbb{Z}_>$

$$\left|\tfrac{1-k}{2}, \tfrac{1-\bar k}{2}\right\rangle$$

$(\bar L_{-1})^{\bar k}$

$h \mapsto 1-h$

Weyl refl.



*In CCFT we can tune Δ to these values.

Now let us turn to the OPE $\Longleftrightarrow$ collinear limits of scattering

$$\lim_{z_{ij}\to 0} A_n(p_1,\dots,p_n) \to \sum_{s\in\pm 2} \mathrm{Split}^{s}_{s_i s_j}(p_i,p_j)\, A_{n-1}(P=p_i+p_j)$$

for the graviton we have

$$\mathrm{Split}^{2}_{22} = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_p^2}{\omega_i\omega_j} \qquad \mathrm{Split}^{-2}_{2\,-2} = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_j^3}{\omega_i\omega_p^2}$$

Using our Mellin transform map

$$\int_0^\infty d\omega_i\,\omega_i^{\Delta_i-1}\int_0^\infty d\omega_j\,\omega_j^{\Delta_j-1}\,\mathrm{Split}^{2}_{22}\,(\cdot)$$

$$= -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\left[\int_0^1 dt\, t^{\Delta_i-1}(1-t)^{\Delta_j-2}\right]\int_0^\infty d\omega_p\,\omega_p^{\Delta_i+\Delta_j-1}\,(\cdot)$$

where $\omega_i = t\,\omega_p \quad \omega_j=(1-t)\omega_p$

The collinear splitting function gives the following OPE

$$\mathcal{O}_{\Delta_1,2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}\, B(\Delta_1-1,\Delta_2-1)\,\mathcal{O}_{\Delta_1+\Delta_2,2}(z_2,\bar{z}_2)$$

Extracting the residues $H^k = \lim_{\varepsilon\to 0}\varepsilon\, G^{+}_{k+\varepsilon}$ $\quad k = 2,1,0,-1,\dots$
we see it closes among the soft currents.

These Δ have primary descendants

$$H^k(z,\bar{z}) = \sum_{n=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_n(z)}{\bar{z}^{\,n+\frac{k-2}{2}}} \qquad w^p_n := \frac{1}{\kappa}(p-n-1)!\,(p+n-1)!\,H^{-2p+4}_n$$

$n=\frac{k-2}{2} \qquad z \qquad 2$

under the radial commutator

$$[A,B](z)=\oint_z \frac{dw}{2\pi i}\, A(w)B(z)$$

we have $[w^p_m, w^q_n]=[m(q-1)-n(p-1)]\,w^{p+q-2}_{m+n}$ a $\wedge Lw_{1+\infty}$ sym!

We've seen that

$$\text{ASG} \longrightarrow \text{angle-dep } \infty \text{ sym enh.} \longrightarrow \text{2D CCFT}$$

taking 2D CCFT seriously

$$\text{collinear splitting} \rightarrow \text{OPE} \rightarrow w_{1+\infty} \text{ sym}$$

we see that the $\Lambda=0$ hologram has an even richer symmetry structure.

by joining forces with the twistor, carroll, amplitudes, & bootstrap folks we can start to exploit it!